

THE COMBINATORIAL FORMULA FOR OPEN GRAVITATIONAL DESCENDENTS

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ABSTRACT. In recent works, [20, 21], descendent integrals on the moduli space of Riemann surfaces with boundary were defined. It was conjectured in [20] that the generating function of these integrals satisfies the open KdV equations. In this paper we prove a formula of these integrals in terms of sums over weighted graphs. Based on this formula, the conjecture of [20] was proved in [5].

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1. INTRODUCTION

The study of the intersection theory on the moduli space of open Riemann surfaces was recently initiated in [20]. The authors constructed a descendent theory in genus 0 and obtained a complete description of it. In all genera, they conjectured that the generating series of the descendent integrals satisfies the open KdV equations. This conjecture can be considered as an open analog of the famous Witten's conjecture [24].

In [21] a construction of the high genus analog appears. In genus 1 all the descendents are calculated, while for high genus only the open string and dilaton were proved.

In this paper we prove a formula for all the descendent integrals as sum over amplitudes of special Feynman diagrams which we call odd critical nodal ribbon graphs. With this formula one can effectively calculate all the open descendents.

Based on the formula proved here, the conjecture of [20] is proved in [5].

1.1. Some general definitions, conventions and notations.

Notation 1.1. For $l \in \mathbb{N}$ we write $[l] = \{1, 2, \dots, l\}$. The set $[0]$ will denote the empty set. For $a, b \in \mathbb{N}$ with $a \leq b$, we write $[a, b] = \{a + 1, a + 2, \dots, b\}$, $[a, a]$ stands for the empty set.

Throughout this article map $m : A \rightarrow \mathbb{Z}$, from an arbitrary set A which is injective away from $m^{-1}(0)$ will be called a *marking* or a *marking of A* . Given a marking, we shall identify elements of $m^{-1}(\mathbb{Z} \setminus \{0\})$ with their images.

We will encounter many types of graphs in the next sections. Dual graphs, to be defined in Section 2, will be denoted by capital Greek letters. Ribbon graphs, to be defined in Sections 4,5, will be denoted by capital English letters.

Many of the objects in this paper, such as surfaces or graphs, will have natural notions of genus, boundary labels and internal labels. A (g, B, I) -object is an object whose genus is g , the set of boundary labels is B , and the set of internal labels is I .

Given a permutation π on a set S , we write s/π the π -cycle of $s \in S$. For $a \in S/\pi$, write $\pi^{-1}(a)$ for the elements which belong to the cycle a .

We shall sometimes use the shorthand notation \mathbf{y} to denote a sequence $\{y_i\}_{i \in [r]}$, if the sequence we are referring to is understood from context.

Let S be a finite set. A (S, l) -set L is a function $L : S \rightarrow [l]$. We write $S = \text{Dom}(L)$. In case $S = [d]$, we simply write a (d, l) -set. We say that L is a l -set if it the set S is understood from the context.

Given two l -sets, L, L' , we write

$$L' \subseteq L,$$

and say that L' is a subset of L , and write $L' \subseteq L$, if

$$\text{Dom}(L') \subseteq \text{Dom}(L), \text{ and } L|_{\text{Dom}(L')} = L'.$$

In this case we define the l -set $L \setminus L'$ by

$$L \setminus L' : \text{Dom}(L) \setminus \text{Dom}(L') \rightarrow [l], (L \setminus L')(s) = L(s).$$

In case $j \in \text{Dom}(L)$ we write $j \in L$. For $i \in [l]$ we put

$$L_i = L^{-1}(i).$$

1.2. Witten's conjecture.

1.2.1. *Intersection numbers.* Denote by $\mathcal{M}_{g,l}$ the moduli space of compact connected Riemann surfaces with l distinct marked points. P. Deligne and D. Mumford defined a natural compactification of it via stable curves in [7] in 1969. Given g, l , a stable curve is a compact connected complex curve with l marked points and finitely many singularities, all of which are simple nodes. We require the automorphism group of the surface to be finite, and the marked points and nodes are all distinct. The moduli space of stable curves of fixed g, l is denoted $\overline{\mathcal{M}}_{g,l}$. It is known that this space is a non-singular complex orbifold of complex dimension $3g - 3 + l$. For the basic theory the reader is referred to [7, 9].

In his seminal paper [24], E. Witten, motivated by theories of 2-dimensional quantum gravity, initiated new directions in the study of $\overline{\mathcal{M}}_{g,l}$. For each marking index i he considered the tautological line bundles

$$\mathbb{L}_i \rightarrow \overline{\mathcal{M}}_{g,l}$$

whose fiber over a point

$$[\Sigma, z_1, \dots, z_l] \in \overline{\mathcal{M}}_{g,l}$$

is the complex cotangent space $T_{z_i}^* \Sigma$ of Σ at z_i . Let

$$\psi_i \in H^2(\overline{\mathcal{M}}_{g,l}; \mathbb{Q})$$

denote the first Chern class of \mathbb{L}_i , and write

$$(1) \quad \langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \rangle_g^c := \int_{\overline{\mathcal{M}}_{g,l}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_l^{a_l}.$$

The integral on the right-hand side of (1) is well-defined, when the stability condition

$$2g - 2 + l > 0$$

is satisfied, all the a_i 's are non-negative integers, and the dimension constraint

$$3g - 3 + l = \sum_i a_i$$

holds. In all other cases $\langle \prod_{i=1}^l \tau_{a_i} \rangle_g^c$ is defined to be zero. The intersection products (1) are often called *descendent integrals* or *intersection numbers*.

Let t_i (for $i \geq 0$) and u be formal variables, and put

$$\gamma := \sum_{i=0}^{\infty} t_i \tau_i.$$

Let

$$F_g^c(t_0, t_1, \dots) := \sum_{n=0}^{\infty} \frac{\langle \gamma^n \rangle_g^c}{n!}$$

be the generating function of the genus g descendent integrals (1). The bracket $\langle \gamma^n \rangle_g^c$ is defined by the monomial expansion and the multilinearity in the variables t_i . The generating series

$$(2) \quad F^c := \sum_{g=0}^{\infty} u^{2g-2} F_g^c$$

is called the *(closed) free energy*. The exponent $\tau^c := \exp(F^c)$ is called the *(closed) partition function*.

1.2.2. KdV equations. Set $\langle \langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \rangle \rangle^c := \frac{\partial^l F^c}{\partial t_{a_1} \partial t_{a_2} \cdots \partial t_{a_l}}$. Witten's conjecture ([24]) says that the closed partition function τ^c becomes a tau-function of the KdV hierarchy after the change of variables $t_n = (2n+1)!! T_{2n+1}$. In particular, it implies that the closed free energy F^c satisfies the following system of partial differential equations ($n \geq 1$):

$$(2n+1)u^{-2} \langle \langle \tau_n \tau_0^2 \rangle \rangle^c = \langle \langle \tau_{n-1} \tau_0 \rangle \rangle^c \langle \langle \tau_0^3 \rangle \rangle^c + 2 \langle \langle \tau_{n-1} \tau_0^2 \rangle \rangle^c \langle \langle \tau_0^2 \rangle \rangle^c + \frac{1}{4} \langle \langle \tau_{n-1} \tau_0^4 \rangle \rangle^c.$$

These equations are known in mathematical physics as the KdV equations. E. Witten ([24]) proved that the intersection numbers (1) satisfy

the string equation

$$\left\langle \tau_0 \prod_{i=1}^l \tau_{a_i} \right\rangle_g^c = \sum_{j=1}^l \left\langle \tau_{a_j-1} \prod_{i \neq j} \tau_{a_i} \right\rangle_g^c,$$

for $2g - 2 + l > 0$. E. Witten has shown that the KdV equations, together with the string equation actually determine the closed free energy F^c completely. R. Dijkgraaf, E. Verlinde and H. Verlinde ([8]) reformulated an alternative description to Witten's conjecture, in terms of the Virasoro algebra, and they have shown that the two descriptions are equivalent.

1.3. Kontsevich's Proof. Witten's conjecture was proved by M. Kontsevich [15]. Kontsevich's proof [15] of Witten's conjecture consisted of two parts. The first part was to prove a combinatorial formula for the gravitational descendents. Let $\mathcal{R}_{g,n}$ be the set of isomorphism classes of trivalent ribbon graphs of genus g with n marked faces. Denote by $V(G)$ the set of vertices of a graph $G \in \mathcal{R}_{g,n}$. Introduce formal variables λ_i , $i \in [n]$. For an edge $e \in E(G)$, let $\lambda(e) := \frac{1}{\lambda_i + \lambda_j}$, where i and j are the numbers of faces adjacent to e . Then we have

$$(3) \quad \sum_{a_1, \dots, a_n \geq 0} \left\langle \prod_{i=1}^n \tau_{a_i} \right\rangle_g^c \prod_{i=1}^n \frac{(2a_i - 1)!!}{\lambda_i^{2a_i+1}} = \sum_{G \in \mathcal{R}_{g,n}} \frac{2^{|E(G)| - |V(G)|}}{|Aut(G)|} \prod_{e \in E(G)} \lambda(e).$$

The second step of Kontsevich's proof was to translate the combinatorial formula into a matrix integral. Then, by using non-trivial analytical tools and the theory of the KdV hierarchy, he was able to prove that F^c satisfies the KdV equations (1.2.2). Other proofs for Witten's conjecture were given, see for example [18, 19].

1.4. Open intersection numbers and the open KdV equations.

1.4.1. Open intersection numbers. In [20] R. Pandharipande, J. Solomon and the author constructed an intersection theory on the moduli space of stable marked disks. Let $\overline{\mathcal{M}}_{0,k,l}$ be the moduli space of stable marked disks with k boundary marked points and l internal marked points. This space carries a natural structure of a compact smooth oriented manifold with corners. One can easily define the tautological line bundles \mathbb{L}_i , for an internal marking i , as in the closed case.

In order to define gravitational descendents, we must specify boundary conditions. The main construction in [20] is a construction of boundary conditions for $\mathbb{L}_i \rightarrow \overline{\mathcal{M}}_{0,k,l}$. In [20], vector spaces $\mathcal{S}_i = \mathcal{S}_{i,0,k,l}$

of *multisections* of $\mathbb{L}_i \rightarrow \partial \overline{\mathcal{M}}_{0,k,l}$, which satisfy the following requirements, were defined. Suppose a_1, \dots, a_l are non-negative integers with $2 \sum_i a_i = \dim_{\mathbb{R}} \overline{\mathcal{M}}_{0,k,l} = k + 2l - 3$, then

- (a) For any generic choice of multisections $s_{ij} \in \mathcal{S}_i$, for $1 \leq j \leq a_i$, the multisection

$$s = \bigoplus_{\substack{i \in [l] \\ 1 \leq j \leq a_i}} s_{ij}$$

vanishes nowhere on $\partial \overline{\mathcal{M}}_{0,k,l}$.

- (b) For any two such choices s and s' we have

$$\int_{\overline{\mathcal{M}}_{0,k,l}} e(E, s) = \int_{\overline{\mathcal{M}}_{0,k,l}} e(E, s'),$$

where $E = \bigoplus_i \mathbb{L}_i^{a_i}$, and $e(E, s)$ is the relative Euler class.

The multisections s_{ij} , as above, are called *canonical*. With this construction the open gravitational descendents in genus 0 are defined by

$$(4) \quad \langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \sigma^k \rangle_0^o := 2^{-\frac{k-1}{2}} \int_{\overline{\mathcal{M}}_{0,k,l}} e(E, s),$$

where E is as above and s is canonical.

In a forthcoming paper [21], J. Solomon and R.T. define a generalization for all genera. Suppose g, k, l are such that

$$(5) \quad 2g - 2 + k + 2l > 0, 2|g + k - 1.$$

In [21] a moduli space $\overline{\mathcal{M}}_{g,k,l}$ which classifies real stable curves with some extra structure is constructed. The moduli space $\overline{\mathcal{M}}_{g,k,l}$ is a smooth oriented compact orbifold with corners, of real dimension

$$(6) \quad 3g - 3 + k + 2l.$$

Note that naively, without adding an extra structure, the moduli of real stable curves of positive genus is non-orientable.

Again, on $\overline{\mathcal{M}}_{g,k,l}$ one defines vector spaces $\mathcal{S}_i = \mathcal{S}_{i,g,k,l}$, for $i \in [l]$, for which the genus g analogs of requirements (a),(b) from above hold. Write

$$(7) \quad \langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \sigma^k \rangle_g^o := 2^{-\frac{g+k-1}{2}} \int_{\overline{\mathcal{M}}_{g,k,l}} e(E, s),$$

for the corresponding higher genus descendents. Introduce one more formal variable s . The *open free energy* is the generating function

$$(8) \quad F^o(s, t_0, t_1, \dots; u) := \sum_{g=0}^{\infty} u^{g-1} \sum_{n=0}^{\infty} \frac{\langle \gamma^n \delta^k \rangle_g^o}{n!k!},$$

where $\gamma := \sum_{i \geq 0} t_i \tau_i$, $\delta := s\sigma$, and again we use the monomial expansion and the multilinearity in the variables t_i, s .

The description of $\overline{\mathcal{M}}_{g,k,l}$ for arbitrary genus, as well as a definition of the boundary conditions is given in Section 2. Throughout this article we shall write $\langle \cdots \rangle$ for $\langle \cdots \rangle_g^o$, as closed descendents will not be considered.

1.4.2. *Open KdV.* The following initial condition follows easily from the definitions ([20]):

$$(9) \quad F^o|_{t_{\geq 1}=0} = u^{-1} \frac{s^3}{6} + u^{-1} t_0 s.$$

In [20] the authors conjectured the following equations:

$$(10) \quad \frac{\partial F^o}{\partial t_0} = \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F^o}{\partial t_i} + u^{-1} s,$$

$$(11) \quad \frac{\partial F^o}{\partial t_1} = \sum_{i=0}^{\infty} \frac{2i+1}{3} t_i \frac{\partial F^o}{\partial t_i} + \frac{2}{3} s \frac{\partial F^o}{\partial s} + \frac{1}{2}.$$

They were called the open string and the open dilaton equation correspondingly. These equations were geometrically proved in [20] for $g = 0$, and for all genera in [21].

Put $\langle \langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \sigma^k \rangle \rangle^o := \frac{\partial^{l+k} F^o}{\partial t_{a_1} \partial t_{a_2} \cdots \partial t_{a_l} \partial s^k}$. The main conjecture in [20] is

Conjecture 1 (Open KdV conjecture). *The following system of equations is satisfied:*

$$(12) \quad (2n+1)u^{-1} \langle \langle \tau_n \rangle \rangle^o = u \langle \langle \tau_{n-1} \tau_0 \rangle \rangle^c \langle \langle \tau_0 \rangle \rangle^o - \frac{u}{2} \langle \langle \tau_{n-1} \tau_0^2 \rangle \rangle^c + \\ + 2 \langle \langle \tau_{n-1} \rangle \rangle^o \langle \langle \sigma \rangle \rangle^o + 2 \langle \langle \tau_{n-1} \sigma \rangle \rangle^o, \quad n \geq 1.$$

In [20] equations (12) were called the open KdV equations. It is easy to see that F^o is fully determined by the open KdV equations (12), the initial condition (9) and the closed free energy F^c . They have also conjectured a Virasoro-type conjecture which should fully describe the open descendents. Both conjectures were proved in [20] for $g = 0$. In [4] Buryak has proved the equivalence of the two conjectures.

1.5. **The open combinatorial formula.** A topological open $(g, \mathcal{B}, \mathcal{I})$ -surface with boundary Σ , is a topological connected oriented $(g, \mathcal{B}, \mathcal{I})$ -surface with non-empty boundary. By genus we mean, the genus of the doubled surface obtained by gluing two copies of Σ along $\partial \Sigma$. We define a topological open nodal $(g, \mathcal{B}, \mathcal{I})$ -surface similarly.

Definition 1.2. Let g, k, l be non-negative integers which satisfy conditions 5, and let \mathcal{B}, \mathcal{I} be sets with $|\mathcal{B}| = k, |\mathcal{I}| = l$. A $(g, \mathcal{B}, \mathcal{I})$ -smooth trivalent ribbon graph is an embedding $\iota : G \rightarrow \Sigma$ of a connected graph G into a open $(g, \mathcal{B}, \mathcal{I})$ -surface $(\Sigma, \{x_i\}_{i \in \mathcal{B}}, \{z_i\}_{i \in \mathcal{I}})$ such that

- (a) $\{x_i\}_{i \in \mathcal{B}} \subseteq \iota(V(G))$, where $V(G)$ is the set of vertices of G . We henceforth consider $\{x_i\}$ as vertices.
- (b) The degree of every x_i is 2.
- (c) The degree of any vertex $v \in V(G) \setminus \mathbf{x}$ is 3.
- (d) $\partial\Sigma \subseteq \iota(G)$.
- (e) If $l \geq 1$, then

$$\Sigma \setminus \iota(G) = \coprod_{i \in \mathcal{I}} D_i,$$

where each D_i is a topological open disk, with $z_i \in D_i$. We call the disk D_i the face marked i .

- (f) If $l = 0$, then $\iota(G) = \partial\Sigma$, and $k = 3$. Such a component is called *trivalent ghost*.

The genus $g(G)$ of the graph G is the genus of Σ . The number of the boundary components of G or Σ is denoted by $b(G)$ and $V^I(G)$ stands for the set of internal vertices. Denote by $B(G)$ the set of boundary marked points $\{x_i\}_{i \in \mathcal{B}}$, $I(G) \simeq \mathcal{I}$ is the set of faces.

Definition 1.3. An *odd critical nodal ribbon graph* is $G = (\coprod_i G_i) / N$, where

- (a) $\iota_i : G_i \rightarrow \Sigma_i$ are smooth trivalent ribbon graphs.
- (b) $N \subset (\cup_i V(G_i)) \times (\cup_i V(G_i))$ is a set of *ordered* pairs of boundary marked points (v_1, v_2) of the G_i 's which we identify. After the identification of the vertices v_1 and v_2 the corresponding point in the graph is called a node. The vertex v_1 is called the legal side of the node and the vertex v_2 is called the illegal side of the node.
- (c) Ghost components do not contain the illegal sides of nodes.
- (d) For any component G_i , any boundary component of it contains an odd number of points which are either marked points or legal sides of nodes.

We require that elements of N are disjoint as sets (without ordering).

The set of edges $E(G)$ is composed of the internal edges of the G_i 's and of the boundary edges. The boundary edges are the boundary segments between successive vertices which are not the illegal sides of nodes. For any boundary edge e we denote by $m(e)$ the number of the illegal sides of nodes lying on it. The boundary marked points of G are the boundary marked points of G_i 's, which are not nodes. The set

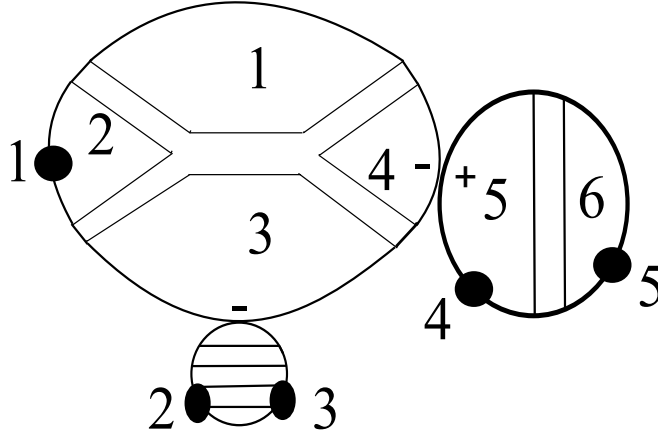


FIGURE 1. A nodal ribbon graph.

of boundary marked points of G will be denoted by $B(G)$, the set of faces by $I(G)$.

An odd critical nodal ribbon graph is naturally embedded into the nodal surface $\Sigma = (\coprod_i \Sigma_i) / N$. The (doubled) genus of Σ is called the genus of the graph. A (g, k, l) -odd critical nodal ribbon graph is a connected odd critical nodal ribbon graph, together with a pair of bijections, $m^B : B(G) \rightarrow [k], m^I : I(G) \rightarrow [l]$, called markings.

Two marked odd critical nodal ribbon graphs $\iota : G \rightarrow \Sigma, \iota' : G' \rightarrow \Sigma'$ are isomorphic, if there is an orientation preserving homeomorphism $\Phi : (\Sigma, \{z_i\}, \{x_i\}) \rightarrow (\Sigma', \{z'_i\}, \{x'_i\})$, of marked surfaces, and an isomorphism of graphs $\phi : G \rightarrow G'$, such that

- (a) $\iota' \circ \phi = \Phi \circ \iota$.
- (b) The maps preserve the markings.

In Figure 1 there is a nodal graph of genus 0, with 5 boundary marked points, 6 internal marked points, three components, one of them is a ghost, two nodes, where a plus sign is drawn next to the legal side of a node and a minus sign is drawn next to the illegal side.

Notation 1.4. Denote by $\tilde{\mathcal{R}}_{g,k,l}^m$ the set of isomorphism classes of odd (g, k, l) -critical nodal ribbon graphs with m legal nodes.

Remark 1.5. In Section 4 we have to consider more general ribbon graphs, and the notions of this subsection are defined in an another equivalent way.

The goal of this paper is to prove the following theorem

Theorem 1.6. Fix $g, k, l \geq 0$ which satisfy conditions 5. Let $\lambda_1, \dots, \lambda_l$ be formal variables. Then we have

$$(13) \quad 2^{\frac{g+k-1}{2}} \sum_{a_1, \dots, a_l \geq 0} \langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_l} \sigma^k \rangle_g^o \prod_{i=1}^l \frac{2^{a_i} (2a_i - 1)!!}{\lambda_i^{2a_i+1}} =$$

$$= \sum_{m \geq 0} \sum_{G = (\coprod_i G_i) / N \in \tilde{\mathcal{R}}_{g,k,l}^m} \frac{\prod_i 2^{|V^I(G_i)| + g(G_i) + b(G_i) - 1}}{|Aut(G)|} \prod_{e \in E(G)} \lambda(e),$$

where

$$\lambda(e) := \begin{cases} \frac{1}{\lambda_i + \lambda_j}, & e \text{ is an internal edge between faces } i \text{ and } j; \\ \frac{1}{(m+1)} \binom{2m}{m} \lambda_i^{-2m-1}, & e \text{ is a boundary edge of face } i \text{ and } m(e) = m; \\ 1, & e \text{ is a boundary edge of a ghost.} \end{cases}$$

1.5.1. *Examples.* $\langle \tau_1 \tau_0 \sigma \rangle_0 = 1$. Thus, for $g = 0, k = 1, l = 2$ the left hand side of Equation 13 with $\lambda_1 = \lambda, \lambda_2 = \mu$, is $\frac{2}{\lambda \mu^3} + \frac{2}{\mu \lambda^3}$. The right hand side receives contributions from several graphs, see Figure 2, (a). The two non nodal contributions in the first line are $\frac{1}{\lambda(\lambda+\mu)\mu^2} + \frac{1}{\mu(\lambda+\mu)\lambda^2}$. The two non nodal contributions in the second line are $\frac{2}{2\lambda^3(\lambda+\mu)} + \frac{2}{2\mu^3(\lambda+\mu)}$. The nodal ones sum to $\frac{1}{\lambda \mu^3} + \frac{1}{\mu \lambda^3}$. And the two sides agree.

The second example is of $\langle \tau_1 \rangle_1 = \frac{1}{2}$. Consider case (b) in Figure 2. The left hand side is $\frac{1}{\lambda^3}$. Non nodal terms do not contribute, as the single relevant graph (leftmost graph of Example b) is not odd. The nodal contribution is exactly $\frac{1}{\lambda^3}$.

The last example (c), is of $\langle \tau_2 \sigma^5 \rangle = 8$. The left hand side gives $\frac{384}{\lambda^5}$. 24 non nodal diagrams, one for each cyclic order of the boundary points, contribute $\frac{24}{\lambda^5}$. There are 120 diagrams with a single node, one for each order, each contributes $\frac{1}{\lambda^5}$. There are 120 diagrams with two nodes, each contribute $\frac{2}{\lambda^5}$, where 2 comes from the Catalan term.

1.6. Proof of the conjecture and related works. In [5] the open KdV conjecture has been proved. The combinatorial formula presented here played a key role in the proof. First, the formula was transformed to a formula in terms of matrix integrals, and then, by analytical tools and ideas from the theory of integrable hierarchies, the integral was shown to be a solution of the open KdV hierarchy.

In [4] it was shown that the open generating function is in fact a wave function of the KdV hierarchy. In [3] a more general generating function, which is a tau-function for the Burgers-KdV system, was introduced. It was conjectured that this function should correspond to an

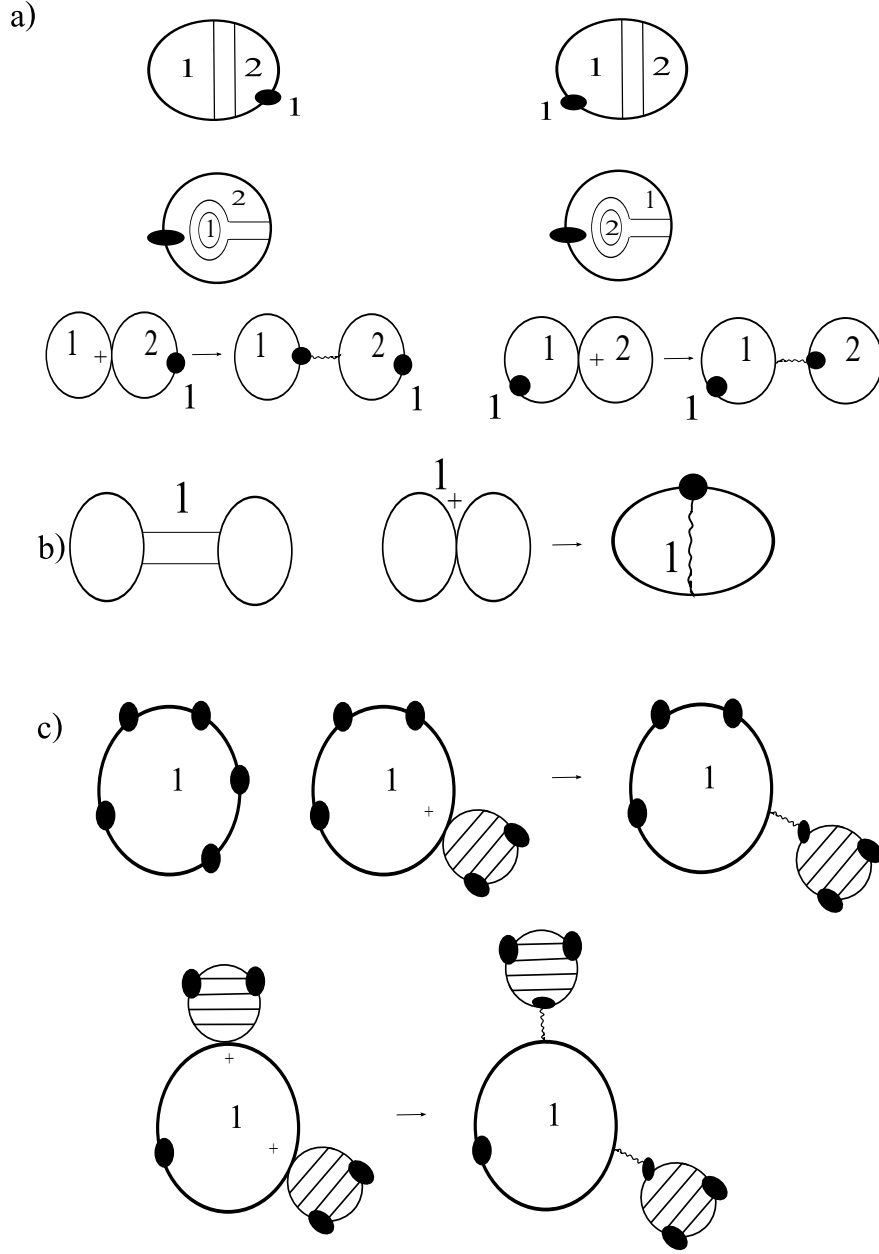


FIGURE 2. Examples of contributing graphs.

open intersection theory which allows adding descendents to boundary marked points. Such a theory is constructed in [22], and there, based on [5] it is shown to satisfy the Burgers-KdV hierarchy.

An alternative description of the open generating function in terms of matrix integrals was found algebraically by A. Alexandrov in [1].

Open problem 1. *Is there a direct geometric way to derive Alexandrov's matrix model?*

1.7. Plan of the paper. In Section 2 the constructions of [21],[20] are reviewed. In particular, graded spin surfaces are defined, as well as their moduli space $\overline{\mathcal{M}}_{g,k,l}$, tautological line bundles and special canonical boundary conditions. Having these in hand, the open gravitational descendents are defined.

In section 3 the notions of sphere bundles and angular forms are reviewed. We write a formula for calculating the integral of the relative Euler class, relative to nowhere vanishing boundary conditions. The main result of this section is a formula for a representative of the angular form of a sphere bundle.

Section 4 is devoted to an open analog of Strebel's stratification. We define symmetric stable Jenkins-Strebel differentials and use them to stratify the moduli space of open surfaces, and then the moduli of graded surfaces. In addition we construct combinatorial sphere bundles. We then show that special canonical multisections are pulled back from the combinatorial moduli. The main result of this section is that the descendents can be calculated on the combinatorial moduli.

Section 5 describes in more details the cells in the stratification which will eventually contribute to the open descendents. We define the notion of an extended Kasteleyn orientation, and show that equivalence class of these are equivalent to the data of a graded spin structure. We use the Kasteleyn orientations to give a more explicit description of the contributing cells, of the boundary conditions and of the orientation of the moduli. As a byproduct we give an alternative proof that the moduli is oriented.

The last section, 6, proves the combinatorial formula, Equation 13. With the aid of the explicit angular form constructed in Section 3 we write an integral which describes the descendent. The integral depends explicitly on the boundary conditions. We then use the properties of special canonical multisections to iteratively integrate by parts, until we reach an integrated form of the formula, Theorem 6.12. Then, by performing a detailed study of the Kasteleyn orientations and multiplicative constants they contribute, we are able to Laplace transform the integrated formula and obtain the main theorem, Theorem 1.6.

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2. THE MODULI, BUNDLES AND INTERSECTION NUMBERS

Throughout this paper we consider orbifolds with corners. See [12, 13] for definitions. This section briefly summarizes some definitions and results of [20, 21].

2.1. Open surfaces.

2.1.1. *Stable open surfaces.* We recall the notion of a *stable marked open surface*.

Definition 2.1. We define a *smooth pointed surface* to be a triple

$$(\Sigma, \mathbf{x}, \mathbf{z}) = (\Sigma, \{x_i\}_{i \in \mathcal{B}}, \{z_i\}_{i \in \mathcal{I}})$$

where

- (a) Σ is a Riemann surface, possibly with boundary.
- (b) An injection $i \in \mathcal{B} \rightarrow x_i \in \partial\Sigma$, where \mathcal{B} is a finite set.
- (c) An injection $i \in \mathcal{I} \rightarrow z_i \in \text{int } \Sigma$, where \mathcal{I} is a finite set.

In case $\partial\Sigma \neq \emptyset$, we say that Σ is an open surface. Otherwise it is closed. We sometimes omit the marked points from our notations. Given a smooth marked surface Σ , we write $B(\Sigma)$ for the set \mathcal{B} , and sometimes also for the set $\{x_i\}_{i \in \mathcal{B}}$. We similarly define $I(\Sigma)$.

A smooth closed pointed surface Σ is called *stable* if

$$2g(\Sigma) + |I(\Sigma)| > 2.$$

A smooth open pointed surface Σ is called *stable* if

$$2g(\Sigma) + |B(\Sigma)| + 2|I(\Sigma)| > 2.$$

Remark 2.2. Σ is canonically oriented, as a Riemann surface. In case $\partial\Sigma \neq \emptyset$, it is endowed with a canonical induced orientation.

Definition 2.3. For a pointed Riemann surface $(\Sigma, \mathbf{x}, \mathbf{z})$ we denote by $(\bar{\Sigma}, \mathbf{x}, \bar{\mathbf{z}})$ the same surface with opposite complex structure. The *doubling* of an open Σ is

$$\Sigma_{\mathbb{C}} = \Sigma \coprod_{\partial\Sigma} \bar{\Sigma},$$

the surface obtained by *Schwartz reflection principle along the boundary* $\partial\Sigma$. For an open Σ we define the *genus* $g(\Sigma)$ to be the genus of $\Sigma_{\mathbb{C}}$. For Σ closed the genus is just the usual genus.

Definition 2.4. A *pre-stable* surface is a tuple

$$\Sigma = (\{\Sigma_{\alpha}\}_{\alpha \in \mathcal{O} \cup \mathcal{C}}, \sim = \sim_B \cup \sim_I, SB)$$

where

- (a) \mathcal{O} and \mathcal{C} are finite sets. For $\alpha \in \mathcal{O}$, Σ_{α} is an open smooth pointed surface; for $\alpha \in \mathcal{C}$, Σ_{α} is a closed smooth pointed surface.
- (b) An equivalence relation \sim_B on $\bigcup_{\alpha} B(\Sigma_{\alpha})$, with equivalence classes of size at most 2. An equivalence relation \sim_I on $\bigcup_{\alpha} I(\Sigma_{\alpha})$, with equivalence classes of size at most 2. We write $B(\Sigma), I(\Sigma)$ for the equivalence classes of size 1 of \sim_B, \sim_I respectively.
- (c) A subset $SB(\Sigma) \subseteq I(\Sigma)$.

Elements of $B(\Sigma)$ are called *boundary marked points*. Elements of $I(\Sigma) \setminus SB(\Sigma)$ are called *internal marked points*. The \sim_B (resp. \sim_I) equivalence classes of size 2 are called boundary (resp. interior) nodes, elements which belong to these equivalence classes are called *half nodes*. Element of SB are called shrunk boundaries. The equivalence classes of $\sim, (\sim_B, \sim_I)$ are collectively called *special (special boundary, special internal) points of Σ* .

We also write $\Sigma = \coprod_{\alpha \in \mathcal{D} \cup \mathcal{S}} \Sigma_{\alpha} / \sim$. If \mathcal{O} is empty and SB is empty, Σ is called a *pre-stable closed surface*. Otherwise it is called a *pre-stable open surface*.

A *pre-stable surface* is *marked*, if in addition it is endowed with markings $m^B : B(\Sigma) \rightarrow \mathbb{Z}$, $m^I : I(\Sigma) \setminus SB \rightarrow \mathbb{Z}$. Write $m = m^I \cup m^B$. Recall that a marking is injective outside of the preimage of 0.

A *pre-stable marked surface* is called a *stable marked surface* if each of its constituent smooth surfaces Σ_{α} is *stable*.

The doubled surface $\Sigma_{\mathbb{C}}$ of a stable open surface is defined as

$$\Sigma_{\mathbb{C}} = \left(\coprod_{\alpha \in \mathcal{O}} (\Sigma_{\alpha})_{\mathbb{C}} \coprod_{\alpha \in \mathcal{C}} \Sigma_{\alpha} \coprod \overline{\Sigma}_{\alpha} \right) / (\sim_B \cup \sim_I \cup \sim_{\bar{I}} \cup \sim_{SB}),$$

where $\sim_{\bar{I}}$ identifies internal marked points of $\{\overline{\Sigma}_{\alpha}\}_{\alpha \in \mathcal{C}}$ if and only if \sim_I identifies the corresponding marked points in $\{\Sigma_{\alpha}\}_{\alpha \in \mathcal{C}}$. \sim_{SB} identifies $z_i \in \Sigma_{\alpha}, \bar{z}_i \in \overline{\Sigma}_{\alpha}$ whenever $i \in SB(\Sigma)$. $\Sigma_{\mathbb{C}}$ is endowed with an involution ϱ , whose fixed point set is $\partial\Sigma$, and such that $\Sigma \simeq \Sigma_{\mathbb{C}} / \varrho$. Write $D(\Sigma) = (\Sigma_{\mathbb{C}}, \varrho)$. We sometimes identify $D(\Sigma), \Sigma_{\mathbb{C}}$. The genus of a stable marked surface Σ is defined to be the usual genus of $\Sigma_{\mathbb{C}}$ as a closed stable surface.

Σ is connected if the underlying space, $\coprod_{\alpha \in \mathcal{D} \cup \mathcal{S}} \Sigma_\alpha / \sim$ is. Σ is smooth if $SB(\Sigma) = \emptyset$, and \sim has only equivalence classes of size 1.

The normalization $Norm(\Sigma)$ of the stable marked surface Σ is the surface $(\{\Sigma_\alpha\}_{\alpha \in \mathcal{O} \cup \mathcal{C}}, \sim', SB', m')$ where \sim' has only size 1 equivalence classes, SB' is empty, and m' agrees with m whenever is defined, and otherwise $m'^I = 0, m'^B = 0$. Whenever a single marked point is marked i , write Σ_i for the component of $Norm(\Sigma)$ which contains marked point z_i .

A topological open stable marked surface is an isotopy class of open stable marked surfaces.

Definition 2.5. An isomorphism between $\Sigma = (\{\Sigma_\alpha\}_{\alpha \in \mathcal{O} \cup \mathcal{C}}, \sim, SB, m)$ and $\Sigma' = (\{\Sigma'_\alpha\}_{\alpha \in \mathcal{O}' \cup \mathcal{C}'}, \sim', SB', m')$ is a tuple $f = (f^\mathcal{O}, f^\mathcal{C}, \{f^\alpha\}_{\alpha \in \mathcal{O} \cup \mathcal{C}})$ such that

- (a) For $\alpha \in \mathcal{O}$, with Σ_α stable, $f^\alpha : \Sigma_\alpha \rightarrow \Sigma'_{f^\mathcal{O}(\alpha)}$ is a biholomorphism which takes marked points to marked points. For $\alpha \in \mathcal{C}$, $f^\alpha : \Sigma_\alpha \rightarrow \Sigma'_{f^\mathcal{C}(\alpha)}$ is a biholomorphism which takes special points to marked points.
- (b) $m' \circ f = m$.
- (c) $f(SB) = SB'$.

We denote by $Aut(\Sigma)$ the group of the automorphisms of Σ .

2.1.2. *Stable graphs.* It is useful to encode some of the combinatorial data of stable marked surfaces in graphs.

Definition 2.6. A (not necessarily connected) *pre-stable dual graph* Γ is a tuple

$$(V = V^O \cup V^C, H = H^B \cup H^I, \sigma_0, \sim = \sim_B \cup \sim_I, g, H^{SB}, m = m^B \cup m^I),$$

where

- (a) V^O, V^C are finite sets called *open and closed vertices, respectively*.
- (b) H^B, H^I are finite sets of boundary and internal half edges.
- (c) $\sigma_0 : H \rightarrow V$ associates any half edge to its vertex.
- (d) \sim_B is an equivalence relation on H^B with equivalence classes of sizes 1 or 2. Denote by T^B the equivalence classes of size 1 of \sim_B . \sim_I is an equivalence relation on H^I with equivalence classes of sizes 1 or 2. Denote by T^I the equivalence classes of size 1 of \sim_I .
- (e) $H^{SB} \subseteq T^I$.
- (f) $g : V \rightarrow \mathbb{Z}_{\geq 0}$ is a genus assignment.
- (g) $m^B : T^B \rightarrow \mathbb{Z}$, $m^I : T^I \setminus H^{SB} \rightarrow \mathbb{Z}$ are markings.

We call T^B *boundary tails*, H^{SB} *shrunk boundaries*, and $T^I \setminus H^{SB}$ *internal tails*. Set $T = T^I \cup T^B$. \sim_B induces a fixed point free involution on $H^B \setminus T^B$. Similarly, \sim_I induces a fixed point free involution on $H^I \setminus T^I$. We denote this involution on $H \setminus T$ by σ_1 . We set $E^B = (H^B \setminus T^B)/\sim_B$, the set of boundary edges. We define $E^I = (H^I \setminus T^I)/\sim_I \cup H^{SB}$. We put $E = E^I \cup E^B$, the set of edges. We denote by σ_0^B the restriction of σ_0 to H^B , in a similar fashion we define σ_0^I .

We require that for all $h \in H^B$, $\sigma_0(h) \in V^O$.

We say that Γ is connected if its underlying graph, (V, E) is connected.

For a vertex v we set $k(v) = |(\sigma_0^B)^{-1}(v)|$. It is defined to be 0 if v is closed. We set $l(v) = |(\sigma_0^I)^{-1}(v)|$. Write $SB(v)$ for the number of shrunk boundaries of v . We define $\varepsilon : V \rightarrow \{1, 2\}$ to be 1 if and only if $v \in V^O$.

The genus of a stable connected dual graph Γ is defined by

$$g(\Gamma) = \sum_{v \in V^O} g(v) + 2 \sum_{v \in V^C} g(v) + |E^B| + 2|E^I| - |H^{SB}| - |V^O| - 2|V^C| + 1.$$

A closed vertex $v \in V^C$ is stable if

$$2g(v) + l(v) > 2.$$

An open vertex $v \in V^O$ is stable if

$$2g(v) + k(v) + 2l(v) > 2.$$

A dual graph Γ is stable if all its vertices are.

$Norm(\Gamma)$, the *normalization* of the graph Γ is the unique stable graph $(V', H', \sigma'_0, \sim', g', H'^{SB}, m')$ with $V' = V, H' = H, \sigma'_0 = \sigma_0, g' = g, H'^{SB} = \emptyset$, and \sim' has only classes of size 1. m' agrees with m , whenever m is defined. Otherwise $m' = 0$.

When $i \in I$ satisfies $|(m^I)^{-1}(i)| = 1$, we denote by $v_i(\Gamma)$ the connected component of $Norm(\Gamma)$ which contains the tails marked i .

A stable dual graph is *effective* if

- (a) Any internal half edge is a tail or a shrunk boundary.
- (b) Any vertex without internal tails has exactly three boundary half edges and genus 0.
- (c) Different vertices without internal half edges are not adjacent.

Definition 2.7. An *isomorphism* between graphs

$$\Gamma = (V, H, \sigma_0, \sim, g, H^{SB}, m), \Gamma' = (V', H', \sigma'_0, \sim', g', H'^{SB}, m')$$

is a pair $f = (f^V, f^H)$ such that

- (a) $f^V : V \rightarrow V'$ is a bijection; $f^H : H \rightarrow H'$ is a bijection.
- (b) $g' \circ f = g$.

- (c) $h_1 \sim h_2 \Leftrightarrow f(h_1) \sim' f(h_2)$.
- (d) $\sigma'_0 = f \circ \sigma_0$.
- (e) $m' \circ f = m$.
- (f) $f(H^{SB}) = H'^{SB}$.

We denote by $Aut(\Gamma)$ the group of the automorphisms of Γ .

We denote by $\mathcal{G}_{g,k,l}^{\mathbb{R}}$ the set of isomorphism classes of all stable graphs of genus g , with

$$Image(m^B) = [k]; Image(m^I) = [l],$$

and let $\mathcal{G}^{\mathbb{R}}$ be the set of isomorphism classes of all stable graphs.

Notation 2.8. Given nonnegative integers k, l with $2g + k + 2l > 2$, denote by $\Gamma_{g,k,l}^{\mathbb{R}}$ the stable graph with $V^O = \{*\}$, $V^C = \emptyset$, with

$$g(*) = g, \quad T^B = H^B \simeq [k], T^I = H^I \simeq [l],$$

where the equivalence is obtained by m^B, m^I , respectively. We similarly define $\Gamma_{g,n}$ as the closed graph with a single vertex of genus g , and $T^I = H^I \simeq [n]$.

To each stable marked genus g surface Σ we associate an isomorphism class of connected stable graph as follows. We set $V^O = \mathcal{O}$ and $V^C = \mathcal{C}$. $H^B = \bigcup_{\alpha} \mathcal{M}_B(\Sigma_{\alpha})$, $H^I = \bigcup_{\alpha} \mathcal{M}_I(\Sigma_{\alpha})$. $H^{SB} = SB(\Sigma)$. The definitions of g, \sim, σ_0, m are straightforward. It is easy to see that although there is a choice in the association of the graph, there is no choice in the association of the isomorphism class. In particular, a tail marked a is associated to a marked point labeled a . An edge between two vertices corresponds to a node between their corresponding components. Note that in fact this correspondence is in the level of topological stable surfaces.

Notation 2.9. The graph associated to a stable surface Σ is denoted by $\Gamma(\Sigma)$.

Note that $Norm(\Gamma(\Sigma)) = \Gamma(Norm(\Sigma))$, and whenever a single internal marked point is marked i , $v_i(\Gamma(\Sigma)) = \Gamma(\Sigma_i)$, when Σ_i is the component of Σ which contains marked point z_i .

Definition 2.10. A surface is called *effective* if it is associated to an effective graph.

2.1.3. Some graph operations.

Definition 2.11. Consider a stable graph Γ . The smoothing of Γ at $f \in E$ is the stable graph

$$d_f \Gamma = \Gamma' = (V', H', \sim', s'_0, g', m')$$

defined as follows. Suppose f is the \sim -equivalence class $\{h_1, h_2\}$, write $\sigma_0(h_1) = v_1, \sigma_0(h_2) = v_2$. The vertex set is given by

$$V' = (V \setminus \{v_1, v_2\}) \cup \{v\}.$$

The new vertex v is closed if and only if both v_1 and v_2 are closed.

$$H' = H \setminus \{h_1, h_2\}.$$

and \sim' is the restriction of \sim to H' . For $h \in \sigma_0^{-1}(\{v_1, v_2\})$ we define $\sigma'_0(h) = v$. Otherwise $\sigma'_0(h) = \sigma_0(h)$. For any tail t , $m'(t) = m(t)$.

$$g'(v) = \begin{cases} g(v_1) + \varepsilon(v_1), & \text{if } v_1 = v_2, \\ g(v_1) + g(v_2), & \text{if } v_1 \neq v_2, \varepsilon(v_1) = \varepsilon(v_2), \\ \varepsilon(v_1)g(v_1) + \varepsilon(v_2)g(v_2), & \text{otherwise.} \end{cases}$$

When $f \in H^{SB}$, a shrunk boundary of vertex v , then $V' = V$, $H' = H \setminus \{f\}$, $H'^{SB} = H^{SB} \setminus \{f\}$. We update \sim', σ'_0, m' as above. $g'(w) = g(w)$, when $w \neq v$. $g'(v) = \varepsilon(v)g(v) + 2 - \varepsilon(v)$.

Observe that there is a natural proper injection $H' \hookrightarrow H$, so we may identify H' with a subset of H . This identification induces identifications of tails and of edges. Using the identifications, we extend the definition of smoothing in the following manner. Given a set $S = \{f_1, \dots, f_n\} \subseteq E(\Gamma)$, define the smoothing at S as

$$d_S \Gamma = d_{f_n} (\dots d_{f_2} (d_{f_1} \Gamma) \dots).$$

Observe that $d_S \Gamma$ does not depend on the order of smoothings performed.

Definition 2.12. A *smoothing* of a stable marked surface Σ in an internal node $z_\nu \sim z_\mu$ is the unique open stable topological surface Σ' , such that there exists a simple closed trajectory $\gamma \hookrightarrow \Sigma'$, and a map $\varphi : \Sigma' \rightarrow \Sigma$ which takes γ to the node, and restricts to an orientation preserving homeomorphism $\varphi : \Sigma' \setminus \gamma \simeq \Sigma \setminus \{z_\mu, z_\nu\}$. In this case we say that γ is contracted to the node. We say that γ *degenerates* to z_ν when this time γ is an *oriented* simple trajectory in Σ' , if γ is contracted to the node, and the φ -preimage of a small enough neighborhood of z_ν lays in the left of γ . The definitions of smoothing in a boundary node, or degeneration to a boundary half node are analogous, only with a simple trajectory that connects two boundary points.

The smoothing of a topological stable surface Σ in a shrunk boundary z_ν is the unique topological stable surface Σ' such that there exists a boundary component $\partial \Sigma'_\nu$, and $\varphi : \Sigma' \rightarrow \Sigma$, such that $\varphi(\partial \Sigma'_\nu) = z_\nu$, and $\varphi : \Sigma' \setminus \partial \Sigma'_\nu \simeq \Sigma \setminus z_\nu$.

Note that if e is the edge of $\Gamma(\Sigma)$ which corresponds to the node $z_\nu \sim z_\mu$ in Σ , then $\Gamma(\Sigma') = d_e \Gamma(\Sigma)$, where Σ' is the smoothing of Σ in that node. Similarly for smoothing in shrunk boundaries.

Note that in case $\Gamma = d_S \Gamma'$, then H' is canonically a subset of H , and we have a natural identification between $E(\Gamma)$ and $E(\Gamma') \setminus S$.

Definition 2.13. We now define boundary maps

$$\partial : \mathcal{G}^{\mathbb{R}} \rightarrow 2^{\mathcal{G}^{\mathbb{R}}}, \quad \partial^! : \mathcal{G}^{\mathbb{R}} \rightarrow 2^{\mathcal{G}^{\mathbb{R}}},$$

by $\partial \Gamma = \{\Gamma' \mid \exists \emptyset \neq S \subseteq E(\Gamma'), \Gamma = d_S \Gamma'\}$, $\partial^! \Gamma = \{\Gamma\} \cup \partial \Gamma$. These maps naturally extend to maps $2^{\mathcal{G}^{\mathbb{R}}} \rightarrow 2^{\mathcal{G}^{\mathbb{R}}}$.

2.1.4. Moduli of open surfaces.

Notation 2.14. For $\Gamma \in \mathcal{G}^{\mathbb{R}}$, denote by $\mathcal{M}_\Gamma^{\mathbb{R}}$ the set of isomorphism classes of stable marked genus g surfaces with associated graph Γ .

Define

$$\overline{\mathcal{M}}_\Gamma^{\mathbb{R}} = \coprod_{\Gamma' \in \partial^! \Gamma} \mathcal{M}_{\Gamma'}^{\mathbb{R}}.$$

We abbreviate $\overline{\mathcal{M}}_{g,k,l}^{\mathbb{R}} = \overline{\mathcal{M}}_{\Gamma_{g,k,l}^{\mathbb{R}}}^{\mathbb{R}}$, $\mathcal{M}_{g,k,l}^{\mathbb{R}} = \mathcal{M}_{\Gamma_{g,k,l}^{\mathbb{R}}}^{\mathbb{R}}$. We similarly define $\overline{\mathcal{M}}_{g,n}$, $\mathcal{M}_{g,n}$.

For $i \in \text{Image}(m^I)$, with $|(m^I)^{-1}(i)| = 1$, write $\mathcal{M}_{v_i(\Gamma)}$ for the moduli of the graph $v_i(\Gamma)$, and denote by $v_i : \mathcal{M}_\Gamma \rightarrow \mathcal{M}_{v_i(\Gamma)}$ the natural map, which on the level of surfaces is just $\Sigma \rightarrow \Sigma_i$.

The space $\overline{\mathcal{M}}_{g,k,l}^{\mathbb{R}}$ is compact smooth orbifold with corners. In general it is non orientable and non connected. Its dimension is

$$\dim_{\mathbb{R}} \overline{\mathcal{M}}_{g,k,l}^{\mathbb{R}} = k + 2l + 3g - 3.$$

A stable marked surface with b boundary nodes or shrunk boundaries belongs to a corner of the moduli space $\overline{\mathcal{M}}_{g,k,l}^{\mathbb{R}}$ of codimension b .

Notation 2.15. Denote by $D : \overline{\mathcal{M}}_{g,k,l}^{\mathbb{R}} \rightarrow \overline{\mathcal{M}}_{g,k+2l}^{\mathbb{R}}$ the doubling map $\Sigma \rightarrow \Sigma_{\mathbb{C}}$.

2.2. Graded surfaces. We present here the extra structure needed for the definition of intersection theory for open Riemann surfaces, following [21]. A more detailed study of nodal behaviour and orbifold structures will appear there.

2.2.1. *Smooth graded surfaces.* Let Σ be a smooth genus g open surface. A *real spin structure* twisted in $\{x_i\}_{i \in \mathcal{B}_1}, \{z_i\}_{i \in \mathcal{I}_1}, \mathcal{I}_1 \subseteq \mathcal{I}$, and $\mathcal{B}_1 \subseteq \mathcal{B}$, is a triple $(\mathcal{L}, b, \tilde{\varrho})$, where $\mathcal{L} \rightarrow \Sigma_{\mathbb{C}}$, is a line bundle over the doubled surface $(\Sigma_{\mathbb{C}}, \varrho)$, b is an isomorphism

$$b : \mathcal{L}^{\otimes 2} \simeq \omega_{\Sigma_{\mathbb{C}}}(-\sum_{i \in \mathcal{B}_1} x_i - \sum_{i \in \mathcal{I}_1} z_i + \bar{z}_i),$$

where $\omega_{\Sigma_{\mathbb{C}}}(-\sum_{i \in \mathcal{B}_1} x_i - \sum_{i \in \mathcal{I}_1} z_i + \bar{z}_i)$ is the canonical bundle twisted in $\{x_i\}_{i \in \mathcal{B}_1}, \{z_i, \bar{z}_i\}_{i \in \mathcal{I}_1}$. $\tilde{\varrho} : \mathcal{L} \rightarrow \mathcal{L}$, is an involution which lifts $d\varrho$, the induced involution on $\omega_{\Sigma_{\mathbb{C}}}$.

$\tilde{\varrho}, d\varrho$ restrict to conjugations on the fibers of

$$\mathcal{L} \rightarrow \partial\Sigma_{\mathbb{C}}, \omega_{\Sigma_{\mathbb{C}}}(-\sum_{i \in \mathcal{B}_1} x_i) \rightarrow \partial\Sigma_{\mathbb{C}}.$$

These conjugations define a real subbundle which is invariant under ϱ . For $\omega(-\sum_{i \in \mathcal{B}_1} x_i)_{\Sigma_{\mathbb{C}}}^{\varrho} \rightarrow \partial\Sigma_{\mathbb{C}}$, this real line bundle is oriented. Indeed, take any nowhere vanishing section $\xi \in \Gamma(T\partial\Sigma_{\mathbb{C}} \rightarrow \partial\Sigma_{\mathbb{C}})$, which points in the direction of the orientation on $\partial\Sigma_{\mathbb{C}}$. The orientation of $\omega_{\Sigma_{\mathbb{C}}}^{\varrho}|_{\partial\Sigma \setminus i \in \mathcal{B}_1}$, is defined by a section $\hat{\xi}$ which satisfies $\hat{\xi}(\xi) > 0$. Such a section is said to be *positive*. Thus, using b , it is seen that for any connected component of $\partial\Sigma_{\mathbb{C}} \setminus \{x_i\}_{i \in \mathcal{B}_1}$, either $\hat{\xi}$ has a root in $\mathcal{L}^{\tilde{\varrho}}$, or $-\hat{\xi}$. If for each connected component of $\partial\Sigma_{\mathbb{C}} \setminus A$, where $A \supseteq \{x_i\}_{i \in \mathcal{B}_1}$ is a finite set of points, the positive sections have roots, we say that $(\mathcal{L}, \tilde{\varrho})$ is *compatible* away from A . In case $A = \{x_i\}_{i \in \mathcal{B}_1}$ we say that the structure is compatible.

Proposition 2.16. *If $\mathcal{B}_1 \neq \emptyset$ then there are no compatible real twisted spin structures.*

Proof. Suppose $i \in \mathcal{B}_1$. Let U be a small ϱ -invariant neighborhood of x_i , which contains no other marked points. One can find a ϱ -invariant section $s \in \Gamma(\mathcal{L} \rightarrow U)$, which vanishes nowhere in U , possibly after replacing U by a smaller neighborhood. In ϱ -invariant local coordinates around x_i , the real section zdz generates $\omega_{\Sigma_{\mathbb{C}}}(U)$. Write $f(z) = zdz/b(s^{\otimes 2})$, this is a nowhere vanishing holomorphic function in U . Moreover, f is conjugation invariant, and hence real on U^{ϱ} . In particular, it does not change sign there. But this is impossible for a compatible structure since zdz is positive on exactly one component of $U^{\varrho} \setminus \{x_i\}$. \square

Given a compatible real spin structure, a *lifting* of the spin structure is a choice of a section in

$$\Gamma(S^0(\mathcal{L}^{\tilde{\varrho}}) \rightarrow \partial\Sigma_{\mathbb{C}} \setminus \{x_i\}_{i \in \mathcal{B}}),$$

where S^0 stands for the rank zero sphere bundle. We say that the lifting *alternates* in x_j , and that x_j is a *legal* point, if this choice cannot be extended to $\Gamma(S^0(\mathcal{L}^{\tilde{e}}) \rightarrow \partial\Sigma_{\mathbb{C}} \setminus \{x_i\}_{i \in \mathcal{B} \setminus \{j\}})$. Otherwise the lifting does not alternate in x_j and x_j is an *illegal* point.

Definition 2.17. A *twisted open smooth spin surface* is a smooth surface $(\Sigma, \{x_i\}_{i \in \mathcal{B}}, \{z_i\}_{i \in \mathcal{I}})$, together with a compatible twisted real spin structure twisted in $\{z_i\}_{i \in \mathcal{I}_1}$. In case $\mathcal{I}_1 = \emptyset$, we call it an *open smooth spin surface*. A (twisted) smooth spin surface with a lifting is a (twisted) open spin surface, together with a lifting. A lifting with all boundary points being legal is called a *grading*. A surface with a non twisted spin structure and a grading is called a *graded surface*. An isomorphism of (twisted) spin surfaces with a lifting is an isomorphism of the underlying surfaces and of the line bundles, which respects the involutions, takes the lifting to the lifting in the target, and respects the twistings and alternations.

2.2.2. Stable graded surfaces. We follow the terminology of [10]. Let $\Sigma = \{\Sigma_{\alpha}\}_{\alpha \in \mathcal{C} \cup \mathcal{O}}$ be a stable (g, k, l) -surface. A *real spin structure* twisted in $\{x_i\}_{i \in \mathcal{B}_1}, \{z_i, \bar{z}_i\}_{i \in \mathcal{I}_1}$, $\mathcal{I}_1 \subseteq \mathcal{I}$, and $\mathcal{B}_1 \subseteq \mathcal{B}$, is a triple $(\mathcal{L}, b, \tilde{\rho})$, where $\mathcal{L} \rightarrow \Sigma_{\mathbb{C}}$, is a rank 1 torsion free sheaf over the doubled surface $(\Sigma_{\mathbb{C}}, \varrho)$, b is an homomorphism

$$b : \mathcal{L}^{\otimes 2} \simeq \omega_{\Sigma_{\mathbb{C}}}(-\sum_{i \in \mathcal{B}_1} x_i - \sum_{i \in \mathcal{I}_1} z_i + \bar{z}_i),$$

where $\omega_{\Sigma_{\mathbb{C}}}(-\sum_{i \in \mathcal{B}_1} x_i - \sum_{i \in \mathcal{I}_1} z_i + \bar{z}_i)$ is the dualizing sheaf twisted in $\{x_i\}_{i \in \mathcal{B}_1}, \{z_i, \bar{z}_i\}_{i \in \mathcal{I}_1}$. $\tilde{\rho} : \mathcal{L} \rightarrow \mathcal{L}$, is an involution which lifts $d\varrho$, the induced involution on $\omega_{\Sigma_{\mathbb{C}}}$.

We require

(a)

$$\deg \mathcal{L} = \frac{\deg \omega_{\Sigma_{\mathbb{C}}} - 2|\mathcal{I}_1| - |\mathcal{B}_1|}{2}.$$

(b) b is an isomorphism on the locus where \mathcal{L} is locally free.

(c) For any point p where \mathcal{L} is not free the length of $\text{coker}(b)$ is 1.

In particular, b is an isomorphism away from nodes. Nodes where b is not an isomorphism are called *Neveu-Schwartz*, at these nodes the last requirement says exactly that b vanishes in order 2. The other nodes are called *Ramond*.

Remark 2.18. Suppose Σ is a a nodal curve, z a node with preimages $z_{\nu}, z_{\mu} \in \text{Norm}(\Sigma)$. Then there are natural residue maps $\text{res}_{\eta} : (\text{Norm}^* \omega_{\Sigma})_{z_{\eta}} \simeq \mathbb{C}$. These induce an isomorphism $a : (\text{Norm}^* \omega_{\Sigma})_{z_{\mu}} \simeq (\text{Norm}^* \omega_{\Sigma})_{z_{\nu}}$, by $\text{res}(v) + \text{res}(a(v)) = 0$. In the Ramond case, we also

have an isomorphism $\bar{a} : (Norm^* \mathcal{L})_{z_\mu} \rightarrow (Norm^* \mathcal{L})_{z_\nu}$, and $res(b(v^{\otimes 2})) + res(b(a(v)^{\otimes 2})) = 0$. For more details see [10].

When $z \in \Sigma \subset \Sigma_{\mathbb{C}}$ is a shrunk boundary which is Ramond, $d\varrho, \tilde{\varrho}$ lift to complex anti linear isomorphisms between the fibers of $Norm^* \omega_{\Sigma_{\mathbb{C}}}$, $Norm^* \mathcal{L}$ in z_{\pm} , where z_+ is the preimage of z in $Norm(\Sigma)$, and z_- is the preimage of z in $Norm(\bar{\Sigma})$. By composing with a, \bar{a} we get anti linear involutions on the fibers at z_+ . This defines real lines, denote them by $(\omega_{\Sigma}^{\mathbb{R}})_{z_+}, (\mathcal{L}^{\mathbb{R}})_{z_+}$, together with maps $res : (\omega_{\Sigma}^{\mathbb{R}})_{z_+} \simeq \mathbb{R}$, and $b^2 : (\mathcal{L}^{\mathbb{R}})_{z_+} \rightarrow (\omega_{\Sigma}^{\mathbb{R}})_{z_+}$, defined by $b^2(v) = b(v^{\otimes 2})$.

We say that the real spin structure is *compatible in a shrunk boundary* z if z is a Ramond node of $\Sigma_{\mathbb{C}}$ and the image of b^2 is in the positive half line $res^{-1}(\mathbb{R}_{\geq 0})$.

The real spin structure is *compatible* if it is compatible in shrunk boundaries and away of special boundary points. Compatibility away from special points is defined as in the smooth case.

A *lifting* of a compatible real spin structure is a choice of a section

$$s \in \Gamma(S^0(\mathcal{L}^{\bar{\varrho}}) \rightarrow \partial \Sigma_{\mathbb{C}} \setminus (\cup_{\alpha \in \mathcal{O}} B(\Sigma_{\alpha}))),$$

where S^0 stands for the rank zero sphere bundle. The notions of alternations and of legal marked point or a legal half node are as in the smooth case.

Proposition 2.19. *(a) A real spin structure on a stable doubled surface, twisted or not, induces a real spin structure, possibly twisted, on any open component of the normalization and a possibly twisted spin structure on any closed component of it. For any node of Σ , the induced structure is either twisted in both of its preimages in the normalization, or not twisted in both. The former case is the Ramond case, the latter is Neveu-Schwartz. If there are no Ramond nodes then the spin structures on the normalization determines the real spin structure on $\Sigma_{\mathbb{C}}$.*

(b) If the real spin structure is compatible, then so is the induced structure on normalization. In this case, in particular, there are no twists in boundary marked points, and no boundary Ramond nodes. In case there are no Ramond internal nodes, but there may be shrunk boundaries, compatible spin structures on the normalization determine the compatible spin structure on $\Sigma_{\mathbb{C}}$.

(c) A lifting induces a lifting on the normalization and vice versa.

Proof. The fact that the twisted spin structure induces one on the normalization, and is induced by one when there are no Ramond nodes is already true in the closed case, see for example [10]. Moreover, it is shown there that given the structures on the normalization and the

identifications of the fibers in preimages of nodes, see Remark 2.18, the structure on the surface is determined. The involution extends uniquely by continuity.

The second claim follows from the fact that one can examine compatibility away from special points. Ramond boundary nodes can not appear by Proposition 2.16. If z is a shrunk boundary, there is a single, up to minus, possible identification map \bar{a} , as in Remark 2.18. Now, if \bar{a} makes the shrunk boundary compatible, $-\bar{a}$ will make it not compatible, and vice versa. The fact that a lifting induces a lifting on the normalization is evident. \square

Definition 2.20. A *twisted open stable spin surface* is a stable surface $(\Sigma, \{x_i\}_{i \in \mathcal{B}}, \{z_i\}_{i \in \mathcal{I}})$, together with a compatible real spin structure twisted in $\{z_i\}_{i \in \mathcal{I}_1}$. In case $\mathcal{I}_1 = \emptyset$, we call it a *stable open spin surface*. A (twisted) stable spin surface with a lifting is a (twisted) open spin surface, together with a lifting such that for any boundary node, exactly one half node is legal. If all the boundary marked points are legal, the twisting is called a *grading*. A *stable graded surface* is a non-twisted stable spin surface with a grading. It is *effective* if the underlying surface is, and in any component of genus 0, 3 special boundary points and no special internal points, its half nodes are legal.

An isomorphism of (twisted) spin surfaces with a lifting is an isomorphism of the underlying surfaces and of the line bundles, which respects the involutions, takes the lifting to the lifting in the target, and respects the twistings and alternations.

Notation 2.21. Denote by $Spin(\Sigma)$ the set of graded spin structures on a stable open surface Σ .

The definition of graded surfaces, together with Proposition 2.19, yield the corollary

Corollary 2.22. *If Σ has no internal nodes, there is a bijection between graded spin structures of Σ and spin structures with a lifting on $Norm(\Sigma)$, such that any marked point of Σ is legal as a point of $Norm(\Sigma)$, and for any node of Σ exactly one half node in $Norm(\Sigma)$ is legal.*

2.2.3. An alternative definition for the smooth case. In this subsection we provide an alternative definition for smooth spin surfaces with a lifting. This definition will be easier to work with. Let $(\Sigma, \{x_i\}_{i \in \mathcal{B}}, \{z_j\}_{j \in \mathcal{I}})$ be a smooth, open or closed, pointed Riemann surface.

Notation 2.23. Denote by $T^1\Sigma$ the S^1 -bundle of $T\Sigma$. For a smooth trajectory $\gamma \subset \Sigma$ we denote the S^0 -bundle of $T\gamma$ by $T^1\gamma$.

When the trajectory γ is oriented, $T^1\gamma$ will stand for length 1 oriented tangent vector field to γ . In particular we shall use the notation $T^1\partial\Sigma$ for the branch of $T^1\partial\Sigma$ which covers the direction of the induced orientation on the boundary.

Remark 2.24. We identify $T^1\Sigma$ as a S^1 -subbundle of length 1 vectors of $T\Sigma$. Similarly for $T^1\gamma$. With this identification we may also identify $T^1\gamma$ as a S^0 -subbundle of $T^1\Sigma|_\gamma$. In what follows we use these identifications without mentioning a choice of metric.

Notation 2.25. For a point $p \in \Sigma$, a vector $w \in T_p\Sigma$, and an angle $\theta \in \mathbb{R}/2\pi\mathbb{R}$, let $r_\theta w = r_\theta(p)w$, be the operator of rotation by θ in the counterclockwise direction. We shall omit p from the notation when it is clear from context. The operator $r_\theta(p)$ is induced on T^1p , and we shall use the same notation.

If u, w are two tangent vectors at p denote the counter clockwise angle from u to w by $\angle(u, w)$.

For a smooth trajectory $\gamma : [0, 1] \rightarrow \Sigma$, there exists a canonical trivialization $\varsigma : [0, 1] \times S^1 \rightarrow T^1\Sigma|_\gamma$, defined by

$$\varsigma(t, \theta) = (\gamma(t), e^{i\theta} v_t), \quad v_t = (T^1)_{\gamma(t)}\gamma.$$

This trivialization defines a continuous family of maps

$$\{p(\gamma)_s^t : T_{\gamma(s)}^1 \rightarrow T_{\gamma(t)}^1\}_{0 \leq s, t \leq 1},$$

uniquely determined by the condition

$$p_2(\varsigma^{-1}(\gamma(s), v)) = p_2(\varsigma^{-1}(\gamma(t), p(\gamma)_s^t v)),$$

where p_2 is the projection on the second coordinate. One can extend the trivialization to the piecewise smooth context by approximation. In case $s = 0, t = 1$ we omit them from the notation and write $p(\gamma)$. One can easily verify, in the piecewise smooth case, that if γ is composed of smooth subtrajectories, $\gamma_i : [a_i \rightarrow a_{i+1}] \rightarrow \Sigma$, where $a_0 = 0 < a_1 < \dots < a_n = 1$, and θ_{i+1} is $\angle(\dot{\gamma}_i|_{\gamma_{i+1}(a_{i+1})}, \dot{\gamma}_i|_{\gamma_i(a_{i+1})})$, then

$$p(\gamma) = p(\gamma_{n-1})r(\theta_{n-1})p(\gamma_{n-2}) \dots r(\theta_1)p(\gamma_0).$$

We shall denote such γ by $\gamma_1 \rightarrow \gamma_2 \rightarrow \dots \gamma_n$. For a *closed* piecewise smooth trajectory γ , we slightly change the definition of p to be

$$p(\gamma) = r_{\theta_0}p(\gamma_{n-1})r_{\theta_{n-1}}p(\gamma_{n-2}) \dots r_{\theta_1}p(\gamma_0),$$

and note that this is in fact the identity map. We shall denote such γ by $\gamma_1 \rightarrow \gamma_2 \rightarrow \dots \gamma_n \rightarrow \gamma_1$.

Definition 2.26. A *twisted spin structure* $\mathbb{S} \rightarrow \Sigma \setminus \{z_j\}_{j \in I}$ on a smooth marked Σ is a S^1 -bundle on $\Sigma \setminus \{z_j\}_{j \in I}$ together with a 2-cover bundle map

$$\pi = \pi^{\mathbb{S}} : \mathbb{S} \rightarrow T^1 \Sigma|_{\Sigma \setminus \{z_j\}_{j \in I}}.$$

Notation 2.27. For a point $p \in \Sigma$, a vector $w \in \mathbb{S}_p$, and an angle $\theta \in \mathbb{R}/4\pi\mathbb{R}$, let $R_\theta w = R_\theta(p)w$, be the operator of rotation by θ in the counterclockwise direction. We shall omit p from the notation when it is clear from context.

The *parallel transport* is the unique continuous family of maps

$$\{P(\gamma)_s^t : \mathbb{S}_{\gamma(s)} \rightarrow \mathbb{S}_{\gamma(t)}\}_{0 \leq s, t \leq 1},$$

which covers $\{p(\gamma)_s^t\}$. We shall sometimes call $P(\gamma)_0^1 v$ the *parallel transport of v along γ* , and write it as $P(\gamma)v$.

Remark 2.28. Note that R covers r in the sense that if $\pi(s) = v$, for $s \in \mathbb{S}_p, v \in T^1 p \Sigma$ then

$$\pi(R_\theta(p)s) = r_\theta(p)v = r_{\theta \pmod{2\pi}}(p)v.$$

Observe that $R_\alpha R_\beta = R_{\alpha+\beta}$. In addition, P, R commute:

$$R_\theta P(\gamma)_s^t v = P(\gamma)_s^t R_\theta v.$$

Definition 2.29. A (twisted) spin structure \mathbb{S} is associated with a function

$$q = q^{\mathbb{S}} : H_1(\Sigma \setminus \{z_j\}_{j \in I}, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2,$$

defined as follows. For $x \in H_1(\Sigma \setminus \{z_j\}_{j \in I}, \mathbb{Z}_2)$, take a piecewise smooth connected representative γ . Then $p(\gamma)$ is the identity. Hence $P(\gamma)$ is either the identity or minus the identity. We define $q(x) = q(\gamma)$ to be 1 in the former case, otherwise it is 0.

For any internal marked point z_j , take a small disk D_j which surrounds it and contains no other marked points in its closure. We define the *twist* in z_j to be $q(\partial D_j)$.

The following theorem appears in [11], we shall use it as a black box throughout the article .

Theorem 2.30. q is a well defined function on $H_1(\Sigma \setminus \{z_j\}_{j \in I}, \mathbb{Z}_2)$. For $\alpha, \beta \in H_1(\Sigma \setminus \{z_j\}_{j \in I}, \mathbb{Z}_2)$,

$$q(\alpha + \beta) = q(\alpha) + q(\beta) + \langle \alpha, \beta, \rangle,$$

where $\langle \alpha, \beta, \rangle$ is the Poincaré pairing.

Proposition 2.31. If $\gamma : [0, 1] \rightarrow \Sigma \setminus \{z_j\}_{j \in I}$ is a piecewise smooth closed curve which bounds a contractible domain, then $P(\gamma)_0^1 = R_{2\pi}$. Moreover, suppose Σ is a topological disk, with a piecewise smooth

boundary γ . Let $\mathbb{S} \rightarrow T^1\Sigma|_\gamma$ be a double cover by a S^1 bundle \mathbb{S} . Then \mathbb{S} can be extended to a non-twisted spin structure on Σ if and only if $P(\gamma)_0^1 = R_{2\pi}$. In this case the extension is unique. In particular, the spin structure can be extended to a marked point z_i if and only if its twist is 0, in that case the extension is unique.

The first part follows from Theorem 2.30, by taking $\alpha = \beta = [\gamma]$. The other parts are also simple and will be omitted.

Definition 2.32. Let (Σ, \mathbb{S}) be an open marked Riemann surface, together with a (twisted) spin structure. Suppose $\partial\Sigma \neq \emptyset$. A *lifting* is a choice of a section

$$s : \partial\Sigma \setminus \{z_i\}_{i \in \mathcal{I}} \rightarrow \mathbb{S}|_{\partial\Sigma \setminus \{x_i\}_{i \in \mathcal{B}}}$$

which covers the *oriented* $T^1(\partial\Sigma \setminus \{x_i\}_{i \in \mathcal{B}})$.

For $j \in \mathcal{B}$, suppose $i : (-\frac{1}{2}, \frac{1}{2}) \rightarrow \partial\Sigma$ is a smooth orientation preserving embedding with $i(0) = x_j$, and $x_b \notin i((-\frac{1}{2}, \frac{1}{2}))$, $b \neq j$. In case

$$\lim_{x \rightarrow 0^-} s(x) \neq \lim_{x \rightarrow 0^+} s(x),$$

we say that the structure *alternates* in x_j , and that x_j is a *legal point*. Otherwise x_j is *illegal* and the structure *does not alternate*. We extend the definition of s to the boundary marked points by $s(x) = \lim_{x \rightarrow 0^+} s(x)$.

A smooth spin surface with a lifting $(\Sigma, \{x_i\}_{i \in \mathcal{B}}, \{z_i\}_{i \in \mathcal{I}}, \mathbb{S}, s)$ is a smooth open Riemann surface together with a spin structure and a lifting. A *smooth graded surface* is a smooth spin surface with a lifting, such that all boundary marked points are legal.

The notion of alternation can be generalized in the following manner.

Definition 2.33. A *bridge* is a piecewise smooth simple trajectory which meets the boundary only in its two distinct endpoints $x, y \in \partial\Sigma \setminus \{x_i\}_{i \in \mathcal{B}}$. Suppose we orient the bridge and parameterize it as

$$\gamma : [0, 1] \rightarrow \Sigma, \quad \gamma(0) = x, \quad \gamma(1) = y.$$

Define $Q(\gamma) \in \mathbb{Z}_2$ by the equation

$$(14) \quad R_{2\pi - \alpha_y}(y)P(\gamma)R_{\alpha_x}(x)s(x) = R_{2\pi Q(\gamma)}(y)s(y).$$

where for $w \in \{x, y\}$, $\alpha_w = \angle((T^1)_w \partial\Sigma, (T^1)_w \gamma) \in (0, 2\pi)$.

$Q(\gamma)$ depends on the orientation but not on the parametrization. An oriented bridge with $Q = 1$ is called a *legal side of the bridge*, otherwise it is called an *illegal side*.

Proposition 2.34. Let Σ be a smooth open spin surface with a lifting. Let γ be and denote by $\bar{\gamma}$ the same bridge with opposite orientation. Then $Q(\gamma) + Q(\bar{\gamma}) = 1$. Thus, any bridge has exactly one legal side and exactly one illegal.

Proof. Work with the notations of Definition 2.33. For $w \in \{x, y\}$, α'_w is defined by $\alpha'_w = \angle((T^1)_w \partial \Sigma, (T^1)_w \bar{\gamma})$. Observe that $\alpha'_x = \alpha_x + \pi$, $\alpha'_y = \alpha_y - \pi$. Apply $R_{2\pi Q(\bar{\gamma})}(x)$ to both sides of Equation 14. By Remark 2.28 and Equation 14 for $\bar{\gamma}$ we obtain

$$R_{2\pi - \alpha_y}(y)P(\gamma)R_{\alpha_x + 2\pi - \alpha'_x}(x)P(\bar{\gamma})R_{\alpha'_y}(y)s(y) = R_{2\pi(Q(\gamma) + Q(\bar{\gamma}))}(y)s(y).$$

The left hand side simplifies to $R_\pi P(\bar{\gamma})R_\pi P(\gamma)s(y)$, which by Proposition 2.31, applied to the piecewise smooth closed curve $\gamma \rightarrow \bar{\gamma} \rightarrow \gamma$, is just $R_{2\pi}s(y)$. \square

Proposition 2.35. (a) *Suppose $(\Sigma, \{z_i\}_{i \in \mathcal{I}}, \mathbb{S})$ is a genus g closed spin surface. Suppose that exactly l_1 marked points have twisting 1. Then l_1 is even. For any closed Riemann surface $(\Sigma, \{z_i\}_{i \in \mathcal{I}})$, there exist 2^{2g} distinct non-twisted spin structures on Σ .*
(b) *Suppose $(\Sigma, \{x_i\}_{i \in \mathcal{B}}, \{z_i\}_{i \in \mathcal{I}}, \mathbb{S}, s)$ is a genus g open spin surface with a lifting. Suppose that exactly k_+ of the boundary marked points are legal, and l_1 internal marked points have twisting 1. Then*

$$l_1 = g + 1 + k_+ \pmod{2}.$$

For any $(\Sigma, \{x_i\}_{i \in \mathcal{B}}, \{z_i\}_{i \in \mathcal{I}}) \in \mathcal{M}_{g,k,l}^{\mathbb{R}}$ with $2|g + k + 1$, there exist exactly 2^g graded structures on G .

Proof. For the first claim, let $\{C_i\}$ be a family of non intersecting circles around each marked point. Then $\sum C_i$ is trivial in the homology of $\Sigma \setminus \mathbf{z}$. By Theorem 2.30, $q(\sum C_i) = \sum q(C_i) = 0$. For the number of spin structures see, for example, [10].

Regarding the second claim, let C_i be as above, and for any boundary component $\partial \Sigma_b$, let C_b be a curve surrounding this boundary, disjoint from it, but isotopic to it in $\Sigma \setminus \mathbf{z}$. By the definitions of q, Q one easily sees that $q(C_b)$ is 1 plus the number of legal marked points of $\partial \Sigma_b$. Again $\sum q(C_i) + \sum q(C_b) = 0$, but this sum equals $l_1 + k_+ + b$, where b is the number of boundaries. It is easy to see that $b = g + 1 \pmod{2}$. For the number of graded structures see [21]. We will also obtain it as a byproduct in Subsection 5.1, see the end of Example 5.19. \square

Lemma 2.36. *The definitions of smooth spin surfaces with a lifting, twisted or not, graded or not, given in this subsection are equivalent to the ones given in Subsection 2.2.1.*

Starting with a spin structure \mathcal{L} in the sense of Subsection 2.2.1, \mathbb{S} is just the S^1 -bundle of \mathcal{L}^* , and the lifting is the reduction of the lifting to that bundle. See [21] for more details, and for the proof of equivalence.

2.2.4. *A comment about the alternative definition in the stable case.* In the stable case, by Proposition 2.19, the sheaf \mathcal{L} and the graded data determine the spin structures and liftings on the normalization, hence by Lemma 2.36, determines the data of \mathbb{S}, s for each component. However, it is determined by it, again, using the same lemma and proposition, only when there are no Ramond nodes. Even when there are such nodes, the data of \mathbb{S}, s for each component determines \mathcal{L} and the graded data up to a finite choice, as explained in the proof of Proposition 2.19. Therefore, since working with the S^1 -bundle and its lifting is more convenient, throughout this paper we shall usually write (Σ, \mathbb{S}, s) to indicate a spin structure with a lifting, and leave \mathcal{L} implicit. We shall sometimes even leave \mathbb{S}, s implicit.

2.2.5. *Spin graphs.* It is useful to encode some of the combinatorial data of spin surfaces with a lifting in graphs.

Definition 2.37. A (pre-)stable spin graph Γ with a lifting is a (pre-)stable graph

$$\Gamma = (V, H, \sim = \sim_B \cup \sim_I),$$

together with a twisting map $tw : H^I \rightarrow \mathbb{Z}_2$, and an orientation map $or : H^B \rightarrow \mathbb{Z}_2$. we require

- (a) $tw(h) = tw(\sigma_1(h))$, for any $h \in H^I \setminus T^I$.
- (b) $or(h) + or(\sigma_1(h)) = 1$, for any $h \in H^B \setminus T^B$.
- (c) $\forall h \in H^{SB}, tw(h) = 1$.
- (d) For $v \in V^O$, then

$$\sum_{h \in (\sigma_0^B)^{-1}(v)} or(h) + \sum_{h \in (S_0^I)^{-1}(v)} tw(h) = g + 1 \pmod{2}.$$

- (e) For $v \in V^C$

$$\sum_{h \in \sigma_0^{-1}(v)} tw(h) = 0.$$

A boundary half edge h , and in particular a tail with $or(h) = 0$ is said to be *illegal*, otherwise it is *legal*.

We say that the graph is *stable* if Γ is *stable*.

The normalization $Norm(\Gamma)$ is just the normalization of the underlying graph Γ , with the maps tw, or defined on the tails of $Norm(\Gamma)$ by their values on the corresponding half edges of Γ . Whenever a single internal tail of Γ is marked i , the graph $v_i(\Gamma)$ is the component of $Norm(\Gamma)$ which contains tails i . We call Γ a graded graph if $or(t) = 1$ for all $t \in T^B$, $tw(t) = 0$ for all $t \in T^I$.

A graded Γ is effective if its underlying graph is, and for any $v \in V^O$, with $g(v) = 0$, 3 tails, all in T^B , $or(t) = 1$ for any tail.

Definition 2.38. An *isomorphism* between spin graphs with a lifting (Γ, tw, or) and (Γ', tw', or') is a tuple

$$f = (f^V, f^H)$$

such that

- (a) $f : \Gamma \rightarrow \Gamma'$ is an isomorphism of stable graphs.
- (b) $tw' = tw \circ f^H|_{H^B}$; $or' = or \circ f^H|_{H^I}$.

We denote by $Aut(\Gamma)$ the group of the automorphisms of Γ .

We denote by \mathcal{G} the set of isomorphism classes of all spin graphs with a lifting. We have a natural map

$$\tilde{for}_{spin} : \mathcal{G} \rightarrow \mathcal{G}^{\mathbb{R}}, \quad \tilde{for}_{spin}(\Gamma, tw, or) = \Gamma.$$

Write for_{spin} for its restriction to graded graphs. We denote by $\mathcal{G}_{g,k,l}$ the set of isomorphism classes of graded graphs with $Image(m^B) = [k]$, $Image(m^I) = [l]$. Define $\Gamma_{g,k,l} = for_{spin}^{-1}(\Gamma_{g,k,l}^{\mathbb{R}})$. Thus, this is the unique graph with a single open vertex of genus g , exactly k boundary tails marked by $[k]$, exactly l internal tails marked by $[l]$, $H^{SB} = \emptyset$, $tw \equiv 0$, $or \equiv 1$.

To each graded stable marked surface Σ we associate a graded stable graph (Γ, tw, or) as follows. First, $\Gamma = \Gamma(\Sigma)$. Let $w \in \Sigma_\alpha$ be any special point of this component. It corresponds to some half edge h . If $h \in H^I$, then $tw(h)$ is defined to be the twisting in w . If $h \in H^B$, then $or(h) = 1$ if and only if h is legal. For shortness we denote the graded stable graph corresponding to Σ by $\Gamma(\Sigma)$, omitting tw, or from the notation. Note that $Norm(\Gamma(\Sigma)) = \Gamma(Norm(\Sigma))$, and whenever a single internal marked point is marked i , $v_i(\Gamma(\Sigma)) = \Gamma(\Sigma_i)$.

We can also extend the graph operations to the graded case.

Definition 2.39. The *smoothing* of a stable spin graph with a lifting (Γ, or, tw) , at $f \in E$ is the stable graph

$$d_f \Gamma = (\Gamma', or', tw')$$

such that $d_f(\Gamma) = \Gamma'$. Recall we may identify H' as a subset of H . We define tw', or' as the restrictions of tw, or with respect to this identification. Given a set $S = \{f_1, \dots, f_n\} \subseteq E(\Gamma)$, define the smoothing at S as

$$d_S \Gamma = d_{f_n}(\dots d_{f_2}(d_{f_1} \Gamma) \dots).$$

Note that again in case $\Gamma = d_S \Gamma'$, then H' is canonically identified as a subset of H , and or, tw respect this identification.

Definition 2.40. We now define boundary maps

$$\partial : \mathcal{G} \rightarrow 2^{\mathcal{G}}, \quad \partial^! : \mathcal{G} \rightarrow 2^{\mathcal{G}},$$

by $\partial\Gamma = \{\Gamma' \mid \exists \emptyset \neq S \subseteq E(\Gamma'), \Gamma = d_S \Gamma'\}$, $\partial^! \Gamma = \{\Gamma\} \cup \partial\Gamma$.
Again, these maps extend to maps $2^{\mathcal{G}} \rightarrow 2^{\mathcal{G}}$.

2.2.6. $\overline{\mathcal{M}}_{g,k,l}$.

Notation 2.41. For $\Gamma \in \mathcal{G}$, denote by \mathcal{M}_Γ the set of isomorphism classes of marked genus g spin surfaces with a lifting, associated to graph Γ .

Define

$$\overline{\mathcal{M}}_\Gamma = \coprod_{\Gamma' \in \partial^! \Gamma} \mathcal{M}_{\Gamma'}.$$

Define $\overline{\mathcal{M}}_{g,k,l} = \overline{\mathcal{M}}_{\Gamma_{g,k,l}}$. Similarly define $\mathcal{M}_{g,k,l}$ as the subspace parameterizing smooth surfaces.

For a marking i , denote by $v_i : \mathcal{M}_\Gamma \rightarrow \mathcal{M}_{v_i(\Gamma)}$ the canonical map $\Sigma \rightarrow \Sigma_i$.

The space $\overline{\mathcal{M}}_{g,k,l}$ is a compact smooth orbifold with corners of real dimension $3g - 3 + k + 2l$. It is endowed with a canonical orientation. $\overline{\mathcal{M}}_\Gamma$ is a suborbifold with corners, which is the closure of \mathcal{M}_Γ , for any $\Gamma \in \mathcal{G}_{g,k,l}$. The map For_{spin} is an orbifold branched cover. A graded surface with b boundary nodes belongs to a corner of the moduli space $\overline{\mathcal{M}}_{g,k,l}$ of codimension b . Thus $\partial\overline{\mathcal{M}}_{g,k,l}$ consists of graded stable surfaces with at least one boundary node. For details see [21].

Remark 2.42. Observe that generically, when there are no automorphisms, the set of spin structures or graded spin structures on a surface depend only on its topology.

Remark 2.43. Although a stratum \mathcal{M}_Γ which parameterizes surfaces with shrunk boundary correspond to boundary strata in $\overline{\mathcal{M}}_{g,k,l}^{\mathbb{R}}$, it is not the case when adding a grading. The reason is that there are two strata of full dimension with a codimension 1 boundary is \mathcal{M}_Γ . These strata correspond to the two possible liftings of the spin structure on the boundary $\partial\Sigma_b$ which shrinks to a point in surfaces $\Sigma \in \mathcal{M}_\Gamma$. These two strata are identified, since in our definition we required compatibility in Ramond nodes, but we did not choose a *lifting*.

The *universal curve* $\overline{\mathcal{C}}_{g,k,l} \rightarrow \overline{\mathcal{M}}_{g,k,l}$ is the space whose fiber over $[\Sigma] \in \overline{\mathcal{M}}_{g,k,l}$ is Σ . Its topology can be defined as in the closed case.

The following lemma is useful for understanding the geometry of $\overline{\mathcal{M}}_{g,k,l}$, see [21] for details.

Lemma 2.44. (a) q, Q are isotopy invariants, in the sense that if $(\Sigma_s)_{0 \leq s \leq 1}$ is a path in $\overline{\mathcal{M}}_{g,k,l}$, and $(\gamma_{ts})_{0 \leq s, t \leq 1}$ is a continuous family of simple paths $\gamma_{\cdot, s} \subseteq \Sigma_s \hookrightarrow \overline{\mathcal{C}}_{g,k,l}$, which miss the special

points, and which are either all bridges or all closed. Then in case they are all bridges then $Q(\gamma_{\cdot,s})$ is fixed, for any continuous choice of orientations on $\gamma_{\cdot,s}$, otherwise $q(\gamma_{\cdot,s})$ is fixed.

- (b) Suppose now that $(\Sigma_s)_{0 \leq s \leq 1}$ is a path in $\overline{\mathcal{M}}_{g,k,l}$, and $(\gamma_{ts})_{0 \leq s, t \leq 1}$ is a continuous family of paths $\gamma_{\cdot,s} \subseteq \Sigma_s \hookrightarrow \overline{\mathcal{C}}_{g,k,l}$, which for $s < 1$ are simple and miss the special points, and are either all bridges or all closed. Assume $\gamma_{\cdot,1}$ is a constant path mapped to a node or a shrunk boundary. Then if $\gamma_{\cdot,s}$ are all closed, then the node is internal or a shrunk boundary and its twist is $q(\gamma_{\cdot,s})$, for any $s < 1$. If $\gamma_{\cdot,s}$ are all open, then the node is a boundary node. In this case, the illegal side of the bridges degenerate to the illegal half node, in the sense of Definition 2.12.

In particular, by Proposition 2.34, exactly one of the half nodes of each boundary node is legal.

- (c) Two graded spin structures on Σ , without a Ramond node which give rise to the same pair (q, Q) are isomorphic.

Remark 2.45. A classification of all pairs (q, Q) is given in [21].

Notation 2.46. We denote by \tilde{For}_{spin} the canonical map

$$\tilde{For}_{spin} : \overline{\mathcal{M}}_{\Gamma} \rightarrow \overline{\mathcal{M}}_{for_{spin}(\Gamma)}^{\mathbb{R}}$$

defined by forgetting the spin structure and the lifting. Write For_{spin} for the restriction to graded moduli.

2.3. The line bundles \mathbb{L}_i .

Definition 2.47. Let Γ be a stable graph with a unique internal tail marked i . The line bundle $\mathbb{L}_i \rightarrow \overline{\mathcal{M}}_{\Gamma}^{\mathbb{R}}$ is the line bundle whose fiber at $(\Sigma, \{x_j\}_{j \in \mathcal{B}}, \{z_j\}_{j \in \mathcal{I}}) \in \overline{\mathcal{M}}_{\Gamma}^{\mathbb{R}}$ is $T_{z_i}^* \Sigma$. This bundle can also be defined by pulling back, using the doubling map, from the closed moduli.

Let Γ be a spin graph with a lifting and a unique internal tail marked i . The line bundle $\mathbb{L}_i \rightarrow \overline{\mathcal{M}}_{\Gamma}$ is the line bundle whose fiber at $(\Sigma, \{x_j\}_{j \in \mathcal{B}}, \{z_j\}_{j \in \mathcal{I}}) \in \overline{\mathcal{M}}_{\Gamma}$ is $T_{z_i}^* \Sigma$. Equivalently, this bundle can be defined as the pullback of $\mathbb{L}_i \rightarrow \overline{\mathcal{M}}_{for_{spin}(\Gamma)}^{\mathbb{R}}$ by the map For_{spin} .

2.4. Boundary conditions and intersection numbers. We begin with the simple observation

Observation 2.48. Let (Σ, \mathbb{S}, s) be a smooth marked surface with a spin structure and a lifting, Σ' the marked surface obtained by forgetting points $\{x_b\}_{b \in \mathcal{B}'}$ where \mathcal{B}' is an $Aut(\Sigma, \mathbb{S}, s)$ -invariant subset of illegal boundary marked points. Then \mathbb{S} is canonically a (twisted) spin structure for Σ' , and s canonically extends to a lifting on Σ' . In particular, a marked point is legal for (Σ', \mathbb{S}, s) if and only if it is legal for (Σ, \mathbb{S}, s) .

Definition 2.49. Consider $\Gamma \in \mathcal{G}_{g,k,l}$ and $i \in [l]$, and let $v = i/\sigma_0$ be the vertex of Γ which contains the tail marked i . Denote by $v_i^*(\Gamma)$ the following graph, which will be called *the abstract vertex of i in Γ* , or just the *abstract vertex* for shortness.

- (a) $V(v_i^*(\Gamma)) = \{*\}$, a singleton. It is open if and only if v is.
- (b) $T^I(v_i^*(\Gamma)) = T^I(v)$. Any internal tail of $v_i^*(\Gamma)$ which corresponds to a tail marked by $j \in [l]$ is be marked j , otherwise it is marked 0. The twist of any tail of $v_i^*(\Gamma)$ is the same as the twist of the corresponding tail in considered as a tail of v . $H^{SB} = \emptyset$.
- (c) $T^B(v_i^*(\Gamma)) = \{h \in T^B(v) \mid \text{or}(h) = 1\}$, and all of these boundary tails are marked 0.
- (d) $g(v_i^*(\Gamma)) = g(v)$, $E(v_i^*(\Gamma)) = \emptyset$.

Define the map $\text{for}_{\text{illegal}} : \mathcal{G} \rightarrow \mathcal{G}$, which forgets all tails $t \in T^B$ with $\text{or}(t) = 0$. As a consequence of Observation 2.48, it induces a map in the level of moduli, which will be denoted by $\text{For}_{\text{illegal}}$,

Write $\Phi_{\Gamma,i} = \text{For}_{\text{illegal}} \circ v_i : \mathcal{M}_{\Gamma} \rightarrow \mathcal{M}_{v_i^*(\Gamma)}$.

Remark 2.50. In the level of surfaces, $\Phi_{\Gamma,i}(\Sigma)$, for $\Sigma \in \overline{\mathcal{M}}_{\Gamma}$ is the graded smooth surface obtained from Σ by taking the component of z_i , forgetting all illegal nodes, renaming all remaining special points by 0.

Observation 2.51. For Γ as above, the two bundles, $\mathbb{L}_i \rightarrow \mathcal{M}_{\Gamma}$, and $\Phi_{\Gamma,i}^*(\mathbb{L}_i \rightarrow \mathcal{M}_{v_i^*(\Gamma)})$ are canonically isomorphic.

For a proof, see [20, 21].

Definition 2.52. Suppose $A \subseteq \mathcal{G}_{g,k,l}$ is a collection of graphs with at least one boundary edge. A piecewise smooth multisection s of $\mathbb{L}_i \rightarrow \cup_{\Gamma \in A} \overline{\mathcal{M}}_{\Gamma}$ is called *special canonical* on $\cup_{\Gamma \in A} \overline{\mathcal{M}}_{\Gamma}$ if for all $\Lambda \in \partial \Gamma$

$$s|_{\mathcal{M}_{\Lambda}} = \Phi_{\Lambda,i}^* s^{v_i^*(\Lambda)},$$

for some piecewise smooth multisection $s^{v_i^*(\Lambda)}$ of $\mathbb{L}_i \rightarrow v_i^*(\Lambda)$.

In case $A \subseteq \mathcal{G}_{g,k,l}$ is the collection of all graphs with boundary edges, we say that s as above is *special canonical*.

A multisection $s = \bigoplus_{i \in [l], j \in [a_i]} s_{ij}$, of $\bigoplus_i \mathbb{L}_i^{\oplus a_i}$ is special canonical if each component s_{ij} is special canonical.

The theorem has appeared in [20] in the genus 0 case, and will appear soon in [21] for all genera.

Theorem 2.53. Suppose $a_1, \dots, a_l \geq 0$ are integers which sum to $\frac{k+2l+3g-3}{2}$. Then one can choose multisections $\{s_{ij}\}_{i \in [l], j \in [a_i]}$ such that

- (a) For all i, j s_{ij} is a special canonical multisection of $\mathbb{L}_i \rightarrow \partial \overline{\mathcal{M}}_{g,k,l}$.
- (b) The multisection $s = \bigoplus_{i,j} s_{ij}$ vanishes nowhere.

Moreover, for any two choices $\{s_{ij}\}, \{s'_{ij}\}$ which satisfy the above requirements we have

$$\int_{\overline{\mathcal{M}}_{g,k,l}} e(\bigoplus_i \mathbb{L}_i^{\oplus a_j}, s) = \int_{\overline{\mathcal{M}}_{g,k,l}} e(\bigoplus_i \mathbb{L}_i^{\oplus a_j}, s')$$

where $e(E, s)$ is the relative Euler class of a vector bundle E with respect to boundary conditions s , and $s' = \bigoplus_{i,j} s'_{ij}$.

Remark 2.54. When s is a nowhere vanishing boundary conditions for $E \rightarrow M$, where $rk(E) = \dim(M)$, the relative Euler class $e(E, s) \in H^{top}(M, \partial M)$ is defined. Integrating, or capping with the fundamental class, gives by Poincaré-Lefschetz duality an element of $H_0(M)$, which may be view as the (weighted, signed) number of zeroes of a generic extension of s to the whole orbifold. See the appendix in [20] for details. The relative Euler class can be defined for sphere bundles rather than vector bundles, and the class of a vector bundle E is equal to the class of its associated sphere bundle. We shall use these two forms of Euler form interchangeably throughout the paper.

We can now define open intersection numbers.

Definition 2.55. With the notations of Theorem 2.53, define the open intersection number

$$\langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \rangle := 2^{-\frac{g+k-1}{2}} \int_{\overline{\mathcal{M}}_{g,k,l}} e(\bigoplus_i \mathbb{L}_i^{\oplus a_j}, s),$$

where s is a nowhere vanishing special canonical multisection.

2.5. The orientation of $\overline{\mathcal{M}}_{g,k,l}$. We shall now explain the orientations of the spaces $\overline{\mathcal{M}}_{g,k,l}$.

Definition 2.56. Let M be an oriented manifold with boundary. The induced orientation on ∂M , is defined by the exact sequence

$$0 \rightarrow N \rightarrow T\partial M \rightarrow TM|_{\partial M} \rightarrow 0,$$

where N , the dimension 1 normal bundle of ∂M in M , is oriented by taking the outward normal as positive direction, the orientation of TM is given, and the isomorphism is $\det(N) \otimes \det(T\partial M) \simeq \det(TM)$.

As said before, the spaces $\overline{\mathcal{M}}_{g,k,l}$ were proved to be orientable, and moreover were given canonical orientations. We shall now state a lemma from [21] which characterizes this orientation uniquely.

Lemma 2.57. *There is a unique choice of orientations \mathfrak{o}_Γ , for any graded graph Γ all of whose connected components contain a single vertex, which satisfy the following requirements*

- (a) The 0-dimensional spaces $\overline{\mathcal{M}}_\Gamma$, where Γ is a single vertex of genus 0, with either 3 boundary tails, or 1 boundary tail and 1 internal tail, are oriented positively.
- (b) If $\Gamma = \{\Gamma_1, \dots, \Gamma_r\}$, the connected components, then $\mathbf{o}_\Gamma = \prod_{i=1}^r \mathbf{o}_{\Gamma_i}$.
- (c) Let Γ be a graph with a single boundary edge, e , and Set $\Lambda = d_e \Gamma$. Denote by Γ' the graph obtained by detaching that edge into two tails, and forgetting the tail t with $or(t) = 0$. Note that we have a fibration $\mathcal{M}_\Gamma \rightarrow \mathcal{M}_{\Gamma'}$, whose fiber over the graded surface $\Sigma \in \mathcal{M}_{\Gamma'}$, is naturally identified with $\partial\Sigma \setminus \{x_i\}_{i \in B(\Gamma')}$. Then the induced orientation on \mathcal{M}_Γ as a codimension 1 boundary of $\overline{\mathcal{M}}_\Lambda$, agrees with the orientation on \mathcal{M}_Γ induced by the fibration $\mathcal{M}_\Gamma \rightarrow \mathcal{M}_{\Gamma'}$, where the base is given the orientation $\mathbf{o}_{\Gamma'}$, and the fiber over Σ gets the orientation of $\partial\Sigma$.

Note that the uniqueness is easy, by an inductive argument.

3. SPHERE BUNDLES AND RELATIVE EULER CLASS

Given a rank n complex vector bundle $\pi : E \rightarrow M$, and a metric on it, one can define the sphere bundle $\pi : S = S(E) = S^{2n-1}(E) \rightarrow M$ whose fiber S_p at $p \in M$ is the set of length 1 vectors in E_p , the fiber of E at p , with the induced orientation. One can recover the vector bundle from the sphere bundle by as

$$S \times \mathbb{R}_{\geq 0} / \sim,$$

where $(v, r) \sim (v', r')$ if either $r = r' = 0$, or $v = v', r = r'$. The resulting bundle is called *the linearization* of S , and can be given a linear structure.

Definition 3.1. An *angular form* for E (or for S) is a $2n - 1$ -form Φ on S which satisfies the following two requirements.

- (a) $\int_{S_p} \Phi = 1$, for all $p \in M$.
- (b) $d\Phi = -\pi^* \Omega$ where Ω is some $2n$ -form on M .

The form Ω is a local representative of the top Chern form of $E \rightarrow M$, and will be called the *Euler form* which corresponds to Φ . Denote by Φ also the form on $E \setminus M$, where we identify E and its total space, defined by $P^* \Phi$, where $P : E \setminus M \rightarrow S(E)$, is the map

$$(p, v) \rightarrow (p, v/|v|), \quad p \in M, v \in E \setminus M.$$

It is straight forward that

Observation 3.2. The form $|v|\Phi$ extends to a form on all the total space of E .

The following proposition is well known,

Proposition 3.3. *Let $E \rightarrow M$ be a real oriented rank $2n$ vector bundle on a smooth oriented manifold with boundary M of real dimension $2n$. Write Φ for an angular form, and Ω its corresponding Euler form. Given a nowhere vanishing section $s \in \Gamma(E \rightarrow \partial M)$, one can define the relative Euler class $e(E, s) \in H^{2n}(M, \partial M)$. Then*

$$\int_M e(E, s) = \int_M \Omega + \int_{\partial M} s^* \Phi.$$

See [2], Chapter 11, for further discussion.

Suppose now that $E = \bigoplus_{i=1}^n L_i$, is the sum of n complex line bundles L_i . Choose a metric for E for which the line bundles L_i are pairwise orthogonal. Write α_i for an angular form for $S_i = S(L_i)$, and ω_i for the corresponding Euler form, its curvature. Define the functions

$$r_i : E \rightarrow \mathbb{R},$$

to be the length of the projection of $(p, v) \in E$ to L_i . The sphere bundle can be described as the set of vector which satisfy $\sum r_i^2 = 1$. For convenience, denote by $\omega_i, r_i \alpha_i$ the pull-backs of $\omega_i, r_i \alpha_i$ to the total space of E and of $S(E)$, where for the latter form we use Observation 3.2

As far as we know, the following theorem is new.

Theorem 3.4. *The following form,*

(15)

$$\Phi_L = \sum_{k=0}^{n-1} 2^k k! \sum_{i \in \{1, \dots, n\}} r_i^2 \alpha_i \sum_{I \subseteq \{1, \dots, n\} \setminus \{i\}, |I|=k} \bigwedge_{j \in I} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I \cup \{i\}} \omega_h.$$

is an angular form for E whose corresponding Euler form is $\bigwedge_{i=1}^n \omega_i$.

Remark 3.5. It will be useful to view $\Phi = \Phi_L$ as a multilinear function in the variables $r_i, dr_i, \alpha_i, \omega_i$, $i = 1, \dots, n$.

Proof. Restrict to the fiber (i.e., write $\omega_i = 0$ for all i) to obtain the normalized volume form of the $2n - 1$ -sphere in polar coordinates:

$$2^{n-1}(n-1)! \sum_{i \in \{1, \dots, n\}} r_i^2 \alpha_i \bigwedge_{j \neq i} (r_j dr_j \wedge \alpha_j).$$

Calculating $d\Phi_L$, one gets a telescopic sum which turns out to be equal $\bigwedge \omega_i$. Indeed, write

$$S_{I,i} := 2^k k! r_i^2 \alpha_i \bigwedge_{j \in I} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I \cup \{i\}} \omega_h,$$

the contribution for given I, i , where $k = |I|$. Taking the derivative, as $\omega_i, r_i dr_i$ are closed, only r_i^2 or α_j may contribute. Write

$$\begin{aligned} d_1 S_{I,i} &:= 2^{k+1} k! r_i dr_i \alpha_i \bigwedge_{j \in I} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I \cup \{i\}} \omega_h, \\ d_2 S_{I,i} &:= -2^k k! r_i^2 \omega_i \bigwedge_{j \in I} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I \cup \{i\}} \omega_h, \\ d_{3,l} S_{I,i} &:= -2^k k! r_i^2 \alpha_i r_l dr_l \omega_l \bigwedge_{j \in I \setminus \{l\}} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I \cup \{i\}} \omega_h, \end{aligned}$$

for $l \in I$, we see that

$$dS_{I,i} = d_1 S_{I,i} + d_2 S_{I,i} + \sum_{l \in I} d_{3,l} S_{I,i}.$$

Now, fixing I , one has

$$\begin{aligned} (16) \quad \sum_{i \in I} d_1 S_{I \setminus \{i\}, i} &= k 2^k (k-1)! \bigwedge_{j \in I} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I} \omega_h, \\ \sum_{i \notin I} d_2 S_{I,i} &= - \sum_{i \notin I} 2^k k! r_i^2 \bigwedge_{j \in I} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I} \omega_h \\ (17) \quad &= -(1 - \sum_{i \in I} r_i^2) 2^k k! \bigwedge_{j \in I} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I} \omega_h \\ &= -2^k k! \left(\bigwedge_{j \in I} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I} \omega_h - \right. \\ &\quad \left. - \sum_{i \in I} r_i^3 dr_i \bigwedge_{j \in I \setminus \{i\}} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I} \omega_h \right), \end{aligned}$$

where we have used $\sum r_i^2 = 1$ in the second equality. And, fixing I, i ,

$$\begin{aligned} \sum_{l \notin I \cup \{i\}} d_{3,l} S_{I \cup \{l\}, i} &= - \sum_{l \notin I \cup \{i\}} 2^k k! r_i^2 \alpha_i r_l dr_l \omega_l \bigwedge_{j \in I} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I \cup \{i, l\}} \omega_h \\ &= -2^k k! \sum_{l \in I} r_l dr_l r_i^2 \alpha_i \bigwedge_{j \in I} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I \cup \{i\}} \omega_h \\ (18) \quad &= -2^k k! r_i^3 dr_i \alpha_i \bigwedge_{j \in I} (r_j dr_j \wedge \alpha_j) \bigwedge_{h \notin I \cup \{i\}} \omega_h, \end{aligned}$$

where the identity $\sum r_i dr_i = 0$ was used for the second equality. The last passage follows from noting that except the $l = i$ term, for all other $l \in I$ we will get a monomial with two dr_l terms.

Summing equations 16,17,18, over all the possibilities, the only term which is left uncanceled is $\bigwedge \omega_i$, that is,

$$d\Phi = \sum_{I,i} dS_{I,i} = - \bigwedge \omega_i.$$

As needed. \square

Construction\Notation 1. Suppose $S_1, \dots, S_l \rightarrow M$ are piecewise smooth S^1 bundles over a piecewise smooth orbifold with corners. Denote by $S(S_1, \dots, S_l) \rightarrow M$ the $2l-1$ -sphere bundle on M , whose fibers are $S(S_1, \dots, S_l)_x =$

$$= \{(r_1, P_1, r_2, P_2, \dots, r_l, P_l) | P_i \in (S_i)_x, r_i \geq 0, \sum r_i^2 = 1\} / \sim,$$

where \sim is the equivalence relation generated by

$$(r_1, P_1, \dots, 0, P_i, \dots, r_l, P_l) \sim (r_1, P_1, \dots, 0, P'_i, \dots, r_l, P_l),$$

and with the natural topology.

4. SYMMETRIC JENKINS-STREBEL STRATIFICATION

4.1. JS stratification for the closed moduli.

4.1.1. *JS differential and the induced graph.* In this subsection we briefly describe the stratification of moduli of closed stable curves following [15, 26, 17].

Let Σ be a nodal Riemann surface with $2g - 2 + n \geq 0$. A *quadratic differential* α is a meromorphic section of the tensor square of the cotangent bundle. In a local coordinate z , it can be written as $f(z)dz^2$. The residue of α in $w \in \Sigma$ is the coefficient of $\frac{dz^2}{(z-w)^2}$, in the expansion of α around w .

Let γ be a quadratic differential, and $w \in \Sigma$ a point which is neither a zero nor a pole. In a neighborhood U we can take its unique, up to sign, square root α . This is a 1-form, hence can be integrated along a path. This defines a map

$$g : U \rightarrow \mathbb{C}, g(z) = \int_w^z \alpha,$$

where the integral is taken along any path in U .

A *horizontal trajectory* is the preimage of $\mathbb{R} \subset \mathbb{C}$, and it is a smooth path containing w in its interior. It turns out that the notion of horizontal trajectories can be defined also in the case where w is a zero of order $d \geq -1$, where as usual a zero of order $-m$ is a pole of order m . In this case there are exactly $d + 2$ horizontal rays leaving w . When w is a pole of order 2, if its residue is $-\left(\frac{p}{2\pi}\right)^2$, there is a family of

nonintersecting horizontal trajectories surrounding it, whose union is a topological open disk, punctured at w . Moreover, with respect to the metric defined by $|\sqrt{\gamma}|$, the perimeter of each of these trajectories is p .

Example 4.1. Let Σ be the Riemann sphere. For all $p > 0$,

$$\gamma_p = - \left(\frac{p}{2\pi} \right)^2 \left(\frac{dz}{z} \right)^2,$$

is a quadratic differential, whose only poles are in $0, \infty$ and whose horizontal lines are the sets $|z| = r$, for $r > 0$, whose lengths are indeed p . Their union is an open punctured disk. It should be noted that actually this is the only quadratic differential on the sphere, invariant under the reflection in the unit disk, whose only poles are in $0, \infty$ and are equal.

Definition 4.2. Let $(\Sigma, z_1, \dots, z_n, z_{n+1}, \dots, z_{n+n_0})$ be a marked genus g nodal Riemann surface, with $2g - 2 + n \geq 0$. Let p_1, \dots, p_n be positive reals. Write $p_i = 0$, for $i > n$. A *marked component* is a smooth component of the curve with at least one marked point $z_i, i \in [n]$. The other components are called *unmarked*. A *Jenkins-Strebel differential*, or a *JS-differential* for shortness, is a quadratic differential such that

- (a) γ is holomorphic outside of special points. In nodes it has at most simple poles and in the i^{th} marked point it has a double pole with residue $-(p_i/2\pi)^2$. In particular, if $p_i = 0$ there is at most a simple pole at that point.
- (b) γ vanishes identically on unmarked components.
- (c) Let Σ' be any marked component of Σ . When $p_i \neq 0$, if D_i is the punctured disk which is the union of horizontal trajectories surrounding $z_i \in \Sigma'$, then

$$\bigcup \overline{D_i} = \Sigma'.$$

The following theorem was proved in [23] for the smooth case, the nodal case was treated in [17, 26].

Theorem 4.3. *Given a stable marked surface $(\Sigma, z_1, \dots, z_{n+n_0})$, and \mathbf{p} as above, JS differential exists and is unique.*

Given $(\Sigma, \mathbf{z}), \mathbf{p}$ as above, define the surface with extra structure $\tilde{\Sigma}$, and the map $K_{n_0} : \Sigma \rightarrow \tilde{\Sigma}$ as follows. $\tilde{\Sigma}$ is obtained from Σ by contracting any unmarked component to a point, and attaching any such point its *genus defect* and *marking defect*. The genus defect is the genus of the preimage of the point in Σ , the marking defect is the set of marked points in this preimage, which is labeled by a subset of $[n, n + n_0]$.

The JS differential γ induces a metric graph on $\tilde{\Sigma}$ whose vertices are zeroes of order $d \geq -1$ of γ , including the images of unmarked components, and whose edges are the horizontal trajectories, with their intrinsic length. These embedded graphs can be fully described.

Definition 4.4. A $(g, (n, n_0))$ -stable closed ribbon graph is a graph $G = (V, H, s_0, s_1, g, f)$, where

- (a) V is the set of vertices, H is the set of half edges.
- (b) s_0 is a permutation of the half edges issuing each vertex.
- (c) s_1 is a fixed point free involution of H .
- (d) A map $g : V \rightarrow \mathbb{Z}_{\geq 0}$, called the *genus defect*.
- (e) A map $f : [n, n + n_0] \rightarrow V$, called *marking defect*.

The *faces* of the graph are s_2 -equivalence class of half edges, where $s_2 = s_0^{-1}s_1$. We write $F = H/s_2$. The edges are $E = H/s_1$. The *genus* of G can be defined as follows. Glue disks along the faces to obtain a surface $\tilde{\Sigma}$. The genus of G is the genus of $\tilde{\Sigma}$ plus the sum of genus defects in vertices. We require

- (a) For a vertex v of degree 1 or of degree 2, but such that the assigned permutation is a transposition,

$$g(v) + |f^{-1}(v)| \geq 1.$$

- (b) The genus of the graph is g .
- (c) The number of faces is n .

A *stable metric ribbon graph* is a stable ribbon graph together with a metric

$$\ell : E \rightarrow \mathbb{R}_+.$$

We usually write ℓ_e instead of $\ell(e)$.

A graph is *smooth* if all the vertices' permutations s_0 are cyclic, all genus defects are 0 and all marking defects are of size at most 1. The ribbon graph is connected if the underlying graph is. We define isomorphisms and automorphisms in the expected way. Write $\text{Aut}(G)$ for the automorphism group of G .

Notation 4.5. Throughout this article, given a ribbon graph, possibly with extra structure such as a graded ribbon graph, or a nodal graph, which will be defined later, we shall write $[h]$ for the class of the half edge or the edge h under the action of the automorphism group. We similarly define $[A]$ for a subset of edges or half edges.

Remark 4.6. If $\text{Norm} : \text{Norm}(\Sigma) \rightarrow \Sigma$ is the normalization of Σ , and γ is the JS differential on Σ with prescribed perimeters, then $\text{Norm}^*\gamma$ is a JS differential, hence the unique JS differential, on $\text{Norm}(\Sigma)$, with

the same perimeters, and such that marked points which are preimages of nodes have 0–perimeter.

4.1.2. *Combinatorial moduli.* For a closed stable ribbon graph G , write \mathcal{M}_G for the set of all metrics on G , write $\mathcal{M}_G(\mathbf{p})$ for the set of all such metrics where the i^{th} face has perimeter $p_i > 0$. Note that $\mathcal{M}_G = \mathbb{R}_+^{E(G)}/\text{Aut}(G)$. Write $\overline{\mathcal{M}}_G = \mathbb{R}_{\geq 0}^{E(G)}/\text{Aut}(G)$. We similarly define $\overline{\mathcal{M}}_G(\mathbf{p})$.

For $e \in E(G)$, the edge between vertices v_1, v_2 , define the graph $\partial_e G$, the edge contraction, as follows. Write h_1, h_2 for the two half edge of e . $V(\partial_e G) = V(G) \setminus \{v_1, v_2\} \cup \{v_1 v_2\}$, $H(\partial_e G) = H(G) \setminus \{h_1, h_2\}$. s'_1, g', f' are just s_1, g, f when restricted to vertices and half edges of G . For the new vertex $v = v_1 v_2$, $f'(v) = f(v_1) \cup f(v_2)$, $g'(v) = g(v_1) + g(v_2)$ whenever $v_1 \neq v_2$, otherwise it is $g(v_1) + \delta$, where $\delta = 1$ if h_1, h_2 belong to different s_0 –cycles, or else 0. For any half edge h , $h/s_1 \neq e$, define $s'_2(h)$ to be the first half edge among $s_2(h), s_2^2(h), \dots$, which is not a half edge of e . We then put $s'_0 = s'_1(s'_2)^{-1}$.

If a graph G' is obtained from G by a sequence of edge contractions, we have a canonical map $\mathcal{M}_{G'} \hookrightarrow \overline{\mathcal{M}}_G$. We say that $\mathcal{M}_{G'}$ is a cell of $\overline{\mathcal{M}}_G$. Write $\mathcal{M}_{g,(n,n_0)}^{\text{comb}} = \coprod \mathcal{M}_G$, where the union is taken over smooth closed $(g, (n, n_0))$ ribbon graphs. Write $\overline{\mathcal{M}}_{g,(n,n_0)}^{\text{comb}} = \coprod \overline{\mathcal{M}}_G / \sim = \coprod \mathcal{M}_G$, where the union is taken over all closed stable $(g, (n, n_0))$ ribbon graphs, and \sim is induced by edge contractions. Define $\overline{\mathcal{M}}_{g,(n,n_0)}^{\text{comb}}(\mathbf{p}), \mathcal{M}_{g,(n,n_0)}^{\text{comb}}(\mathbf{p})$ by constraining the perimeters to be p_i . In all cases we define the cell structure using edge contractions.

Set $\text{comb} = \text{comb}_{n_0}$ as the canonical maps

$$\text{comb} : \overline{\mathcal{M}}_{g,n+n_0} \times \mathbb{R}_+^n \rightarrow \overline{\mathcal{M}}_{g,(n,n_0)}^{\text{comb}}, \quad \text{comb}_{\mathbf{p}} : \overline{\mathcal{M}}_{g,n+n_0} \rightarrow \overline{\mathcal{M}}_{g,(n,n_0)}^{\text{comb}}(\mathbf{p}),$$

which sends a stable curve and a set of perimeters to the corresponding graph.

We have, see [15, 17, 26]

Theorem 4.7. *Suppose $n > 0$. The maps $\text{comb}, \text{comb}_{\mathbf{p}}$ are continuous surjections of topological orbifolds. $\text{comb}_{\mathbf{p}}$ takes the fundamental class to a fundamental class. Moreover, the cell complex topology described above is the finest topology with respect to which $\text{comb}^{\mathbb{R}}$ is continuous. The maps are isomorphisms onto their images when restricted to $\mathcal{M}_{g,n+n_0} \times \mathbb{R}_+^n, \mathcal{M}_{g,n+n_0}$.*

More generally, suppose Γ is a closed dual graph with the property that any vertex without a tail marked by $[n]$ is of genus 0, and has exactly 3 half edges, and any two such vertices are not adjacent. Then $\text{comb}, \text{comb}_{\mathbf{p}}$ restricted to $\mathcal{M}_{\Gamma} \times \mathbb{R}_+^n, \mathcal{M}_{\Gamma}$ are isomorphisms onto their image.

4.1.3. Tautological line bundles and associated forms.

Definition 4.8. Suppose $p_i > 0$. Define the space

$$\mathcal{F}_i(\mathbf{p}) \rightarrow \overline{\mathcal{M}}_{g,n}^{comb}(\mathbf{p})$$

as the collection of pairs (G, ℓ, q) , where $(G, \ell) \in \overline{\mathcal{M}}_{g,n}^{comb}(\mathbf{p})$ and q is a boundary point of the i^{th} face. These spaces, glue together to the bundle $\mathcal{F}_i \rightarrow \overline{\mathcal{M}}_{g,n}^{comb}$. Define ϕ_i to be the distance from q to the i^{th} vertex, taken along the arc from q in the counterclockwise direction, so that $0 < \phi_1 < \phi_2 < \dots < p_i$, write $\ell_j = \phi_{j+1} - \phi_j$. Orient the fibers with the clockwise orientation.

Define the following 1-form and 2-form

$$(19) \quad \alpha_i = \sum_j \frac{\ell_j}{p_i} d\left(\frac{\phi_j}{p_i}\right), \omega_i = -d\alpha_i = \sum_{a < b} d\left(\frac{\ell_a}{p_i}\right) \wedge d\left(\frac{\ell_b}{p_i}\right).$$

For later purposes define $\bar{\alpha}_i = p_i^2 \alpha_i$, $\bar{\omega}_i = p_i^2 \omega_i$ and $\bar{\omega} = \sum_i \bar{\omega}_i$.

$\overline{\mathcal{M}}_{g,(n,n_0)}^{comb}$ and the bundles \mathcal{F}_i carry natural piecewise smooth structures. Moreover, [15] says (see also Theorem 5 in [26])

Theorem 4.9. (a) For $i \in [n]$, $comb^* \mathcal{F}_i \simeq S^1(\mathbb{L}_i)$ canonically.
(b) α_i, ω_i are piecewise smooth angular 1-form and Euler 2-form for \mathcal{F}_i .

Remark 4.10. In [15] \mathcal{F}_i was given the opposite orientation and the equivalence was hence to the bundle $S^1(\mathbb{L}_i^*)$, which is canonically $S^1(\mathbb{L}_i)$ with the opposite orientation.

Thus, combined with Theorem 4.7 we see that all descendents may be calculated combinatorially on $\overline{\mathcal{M}}_{g,n}^{comb}$. In fact, all descendents can be calculated as integrals over the highest dimensional cells of $\overline{\mathcal{M}}_{g,n}^{comb}$. These are parameterized by trivalent ribbon graphs.

4.2. JS Stratification for the open moduli.

4.2.1. *Symmetric JS differentials.* Motivated by Definition 4.2 and Example 4.1 we define

Definition 4.11. Let $(\Sigma, \{z_i\}_{i \in \mathcal{I} \cup \mathcal{P}_0}, \{x_i\}_{i \in \mathcal{B}})$ be a stable open marked Riemann surface, $\mathbf{p} = \{p_i\}_{i \in \mathcal{I}}$ a set of positive numbers. A *symmetric JS differential* on Σ is the restriction to Σ of the unique JS differential of $D(\Sigma)$ whose poles at z_i, \bar{z}_i are $-(p_i/2\pi)^2$, when $i \in \mathcal{I}$, and otherwise are 0. We extend the definition to the case $g = 0, \mathcal{I} = [1], \mathcal{P}_0 = \mathcal{B} = \emptyset$, where the differential is defined to be the restriction of the section γ_{p_1} of Example 4.1.

The existence and uniqueness follow from Theorem 4.3 and the discussion in Example 4.1.

As before, the symmetric JS differential defines a cell decomposition of $D(\Sigma)$, in the smooth case, and in general a metric graph embedded in $D(\Sigma)$, the surface obtained from $D(\Sigma)$ by contracting components with no z_i, \bar{z}_i , $i \in \mathcal{I}$, whose complement is a disjoint union of disks. The uniqueness forces the decomposition to be ρ -invariant.

Lemma 4.12. *If Σ_i is a component $D(\tilde{\Sigma})$, then $\partial\Sigma_i$ is the union of (possibly closed) horizontal trajectories. Any boundary point is a zero of the differential of an even order, possibly 0.*

Proof. The case $g = 0, \mathcal{I} = [1], \mathcal{P}_0 = \mathcal{B} = \emptyset$ follows from the discussion in Example 4.1. In other cases, take $p \in \partial\Sigma$. It cannot belong to the disk cell of any z_j , since otherwise it would have belonged to the cell of \bar{z}_j as well. Thus, $\partial\Sigma$ is contained in the one-skeleton of the decomposition. Regarding the behaviour of the differential at boundary points, each boundary marked point has two horizontal trajectories emanating from it in $\partial\Sigma$. If there are also r such trajectories in Σ° , then because of symmetry there are $2r + 2$ horizontal trajectories from it, which means that it is a zero of order $2r \geq 0$. \square

Lemma 4.12 has the following corollary

Corollary 4.13. *Suppose Σ, \mathbf{p} are as above, and γ is the associated symmetric JS differential. Assume that for some $i \in \mathcal{B}$, forgetting x_i makes no component of Σ unstable. Denote by Σ' the resulting surface, and let $\iota : \Sigma' \rightarrow \Sigma$ be the natural map between the surfaces. Then if γ, γ' are the unique JS differentials for Σ, Σ' with the prescribed perimeters, then*

$$\gamma' = \iota^* \gamma.$$

Indeed, both γ, γ' are JS differentials on Σ' , since there is no pole in x_i . Hence they must be equal.

Remark 4.6 has the following consequence

Corollary 4.14. *If $\text{Norm} : \text{Norm}(\Sigma) \rightarrow \Sigma$ is the normalization of Σ , and γ is the JS differential on Σ with prescribed perimeters, then $\text{Norm}^* \gamma$ is the unique JS differential, on $\text{Norm}(\Sigma)$, with the same perimeters, and such that marked points which are preimages of nodes have 0-perimeter.*

Remark 4.15. Later we will show that special canonical boundary conditions are pulled back from the combinatorial moduli we construct. Since their definition involves normalizations of surfaces (or dual graphs)

and forgetful maps, we are forced to consider components of surfaces which contain nodes. These will correspond to marked points with perimeter 0. For this reason we shall allow throughout this section such points, at the cost of more complicated notations.

4.2.2. Open Ribbon graphs.

Notation 4.16. Let I, B be finite sets. Denote by $D(g, I, B)$ the set of isotopy classes of genus g smooth open marked surfaces, with I being the set of internal marked points, B being the set of boundary marked points. Write $D(g, I)$ for the set of isotopy classes of closed smooth oriented genus g surfaces, which is just a singleton.

Definition 4.17. An *open ribbon graph* is a tuple

$$G = (V = V^I \cup V^B, H = H^I \cup H^B, s_0, s_1, f = f^I \cup f^B \cup f^{P_0}, g, d)$$

and where

- (a) V^I is the set of internal vertices, V^B the set of boundary vertices.
- (b) H^B is the set of boundary half edges, H^I is the set of internal half edges; s_1 is a fixed point free involution on H whose equivalence classes are the edges, E . E^B is the set of edges which contain a boundary half edge.
- (c) A permutation s_0 assigned to each vertex. Should be thought as a cyclic order of the half edges issuing each vertex. We write s_0 also for the product of all these permutations.
We denote by \tilde{V} the set of cycles of s_0 . Write \tilde{V}^I for cycles which do not contain boundary half edges. Set $\tilde{V}^B = \tilde{V} \setminus \tilde{V}^I$. Put by $N : \tilde{V} \rightarrow V$ the map which takes a cycle to the vertex which contains its half edges, and let N^{P_0}, N^B be the restrictions to \tilde{V}^I, \tilde{V}^B , respectively.
- (d) A map $f^B : \mathcal{B} \rightarrow V^B$, where \mathcal{B} is a finite set.
- (e) A map $f^{P_0} : \mathcal{P}_0 \rightarrow V$, where \mathcal{P}_0 is a finite set.
- (f) An inclusion $f^I : \mathcal{I} \hookrightarrow H/s_2$, where $s_2 := s_0^{-1}s_1$.
- (g) A map $g : V \rightarrow \mathbb{Z}_{\geq 0}$, called the *genus defect*.
- (h) For any $v \in V^B$, an element

$$d \in D(g(v), (f^{P_0})^{-1}(v) \cup (N^{P_0})^{-1}(v), (f^B)^{-1}(v) \cup (N^B)^{-1}(v)).$$

For any $v \in V^I$, the unique element $d \in D(g(v), (f^{P_0})^{-1}(v) \cup (N^{P_0})^{-1}(v))$. d is called the *topological defect* of v .

Write $\deg(v)$ for the degree of the vertex v . A *closed shrunk component* is a vertex $v \in V^I$ with

$$2g(v) + |(f^{P_0})^{-1}(v)| + |N^{-1}(v)| > 2.$$

Denote their collection by $SC^C(G)$. An *open shrunk component* is a vertex $v \in V^B$ with

$$2(g(v) + |(f^{\mathcal{P}_0})^{-1}(v)| + |(N^{\mathcal{P}_0})^{-1}(v)|) + |(f^{\mathcal{B}})^{-1}(v)| + |(N^{\mathcal{B}})^{-1}(v)| > 2.$$

Denote their collection by $SC^O(G)$.

We have the following requirements.

- (a) Any half edge appears in the permutation s_0 of exactly one vertex.
We define a graph whose vertices are the elements of V and whose half edges are the elements of H . A half edge is connected to a vertex if and only if it appears in the vertex's permutation s_0 .
- (b) $N(\tilde{V}^B) \subseteq V^B$.
- (c) If $h \in H^B$, then $s_1 h \notin H^B$.
- (d) s_2 preserves the partition $H = H^I \cup H^B$. The image of $f^{\mathcal{I}}$ is exactly H^I/s_2 .
- (e) For $v \in V^I$, if the degree of $deg(v) = 1$, or $deg(v) = 2$ but $|N^{-1}(v)| = 1$, then $|(f^{\mathcal{P}_0})^{-1}(v)| + g(v) \geq 1$.
- (f) For $v \in V^B$, if v has at least one boundary edge and $deg(v) = 2$ then $|(f^{\mathcal{P}_0})^{-1}(v)| + |(f^{\mathcal{B}})^{-1}(v)| + g(v) \geq 1$.
- (g) Any vertex of degree 0 is a *shrunk component*.

We call the elements of H^B/s_2 *boundary components*, and the elements of $F = H^I/s_2$ are called *faces*. $b(G) = |H^B/s_2|$ is the number of boundary components. The sets $\mathcal{I}, \mathcal{P}_0, \mathcal{B}$ are called the sets of internal markings, internal markings of perimeter 0, and boundary markings respectively. \mathcal{B} is also denoted by $B(G)$, define $I(G), \mathcal{P}_0(G)$ similarly. An *internal node* is either a shrunk component with at least one edge and no boundary edges, or an internal vertex whose assigned permutation is not transitive. A boundary vertex v without boundary half edges, with no marking defect and such that $g(v) = 0, |N^{-1}(v)| = 1$ is called a *shrunk boundary*. A boundary vertex v which is either a shrunk component with at least one boundary edge, or that whose assigned permutation is not transitive is called a *boundary node*. A *boundary marked point* is an image of $f^{\mathcal{B}}$ which is not a node. An *internal marked point of perimeter 0* is an image of $f^{\mathcal{P}_0}$ which is not a node. A *boundary half node* is a $(N^{\mathcal{B}})^{-1}$ -preimage of a node. Denote their collection by $HN(G)$. A vertex which is either a node or a shrunk component, or the f -image of a unique element in $\mathcal{P}_0 \cup \mathcal{B}$ is called a *special point*.

We write $i(h) = h/s_2$, and $H_i = \{h \in H | i(h) = i\}$.

An *open metric ribbon graph* is an open ribbon graph together with a positive metric $\ell : E \rightarrow \mathbb{R}_+$. We sometimes write $\ell_h, h \in H$ instead of ℓ_{h/s_1} .

Markings of an open ribbon graph are markings,

$$m^I : \mathcal{I} \cup \mathcal{P}_0 \rightarrow \mathbb{Z}, \quad m^B : \mathcal{B} \rightarrow \mathbb{Z},$$

such that $m^I(\mathcal{P}_0) = 0, m^I(\mathcal{I}) \subset \mathbb{Z}_{\neq 0}$. A graph together with a marking is called a *marked graph*.

An isomorphism of marked graphs, and an automorphism of a marked graph are the expected notions. $\text{Aut}(G)$ denotes the group of automorphisms of G . A metric is *generic* if (G, ℓ) has no automorphisms.

A ribbon graph is said to be closed if $V^B = 0$, it is said to be connected if the underlying graph is connected.

Note that an half edge h is canonically oriented away from its base-point h/s_0 . Throughout the paper we identify boundary marked points, which are vertices, with their (unique) preimages in $B(G) = \mathcal{B}$.

Remark 4.18. Here, unlike in the closed case, the genus defect is not enough to classify surfaces with contracted components. In particular, there are several topologies and later also spin structures for given genus and sets of marked points. Although a combinatorial description can be written, for the purposes of this paper it is not needed.

Notation 4.19. By gluing disks along the faces, any open ribbon graph gives rise to a topological open oriented surface Σ_G . This surface is a union of smooth surfaces, identified in a finite number of points. One can easily define its double, $(\Sigma_G)_{\mathbb{C}}$, as in the non topological case.

Definition 4.20. The genus of the open graph G is defined by

$$g(G) := g((\Sigma_G)_{\mathbb{C}}) + \sum_{v \in V^B} g(v) + 2 \sum_{v \in V^I} g(v).$$

The graph is *stable* if $2g - 2 + |\mathcal{B}| + 2(|\mathcal{I}| + |\mathcal{P}_0|) > 0$.

For a stable open surface $(\Sigma, \{z_i\}_{i \in \mathcal{I} \cup \mathcal{P}_0}, \{x_i\}_{i \in \mathcal{B}})$, define the marked components to be components with at least one z_i , $i \in \mathcal{I}$. The other components are unmarked. Define the surface with extra structure $\tilde{\Sigma} = K_{\mathcal{B}, \mathcal{P}_0}(\Sigma)$, and the map $K_{\mathcal{B}, \mathcal{P}_0} : \Sigma \rightarrow \tilde{\Sigma}$ to be the surface obtained by contracting unmarked components to points, and $K_{\mathcal{B}, \mathcal{P}_0}$ is the quotient map. To any point p in $\tilde{\Sigma}$ we associate genus defect, marking defect, and the topological defect which can be defined by the genus, boundary markings and topological type of the surface obtained by smoothing the nodes in $K_{\mathcal{B}, \mathcal{P}_0}^{-1}(p)$.

Remark 4.21. This definition agrees with the one given for closed surfaces, in the sense that one can also define the doubling D of $\tilde{\Sigma}$ in a natural way, and then $D(\tilde{\Sigma}) \simeq D(\Sigma)$.

Definition 4.22. A *ghost* is a ribbon graph without half edges. A *smooth open ribbon graph* is a stable open ribbon graph any such that any connected component of it has no node or shrunk boundary.

An open ribbon graph is *effective* if

- (a) $\mathcal{P}_0 = \emptyset$.
- (b) Any genus defect is 0.
- (c) There are no internal nodes
- (d) Shrunk components or ghost components v must have

$$(N^{\mathcal{P}_0})^{-1}(v) = \emptyset, |(N^{\mathcal{B}})^{-1}(v)| + |(f^{\mathcal{B}})^{-1}(v)| = 3.$$

The graph is *trivalent* if it is effective, has no shrunk boundaries, all vertices which are not special boundary points are trivalent, and for every special boundary point all the s_0 -cycles are of length 2.

A boundary marked point or a boundary half node in a trivalent graph G which is not a ghost is said to *belong to* a face i if its unique internal half edge belongs to that face.

The following proposition is a consequence of Lemma 4.12, and the closed theory, the proof is in the appendix.

Proposition 4.23. *The unique symmetric JS differential of Σ defines a unique metric graph (G, ℓ) embedded in $K_{\mathcal{B}, \mathcal{P}_0}(\Sigma)$. This graph is an open ribbon graph, whose vertices are $K_{\mathcal{B}, \mathcal{P}_0}$ -images of zeroes of the differential, its edges are $K_{\mathcal{B}, \mathcal{P}_0}$ -images of horizontal trajectories. The boundary edges are embedded in the boundary and cover it, and the defects of vertices agree with the defects of their image in $K_{\mathcal{B}, \mathcal{P}_0}(\Sigma)$, in particular boundary nodes go to boundary nodes. Under this embedding the orientation of any half edge $h \in s_1 H^B$ agrees with the orientation induced on $\partial K_{\mathcal{B}, \mathcal{P}_0}(\Sigma)$. Topologically $K_{\mathcal{B}, \mathcal{P}_0}(\Sigma) = \Sigma_G$.*

Moreover, for any stable $(g, \mathcal{B}, \mathcal{I} \cup \mathcal{P}_0)$ -metric graph is the graph associated to some stable open $(g, \mathcal{B}, \mathcal{I} \cup \mathcal{P}_0)$ -surface and a set of perimeters \mathbf{p} . This surface is unique if the graph is smooth or effective.

We sometimes identify the graph with its image under the embedding. In particular, throughout this article we shall consider an edge as a trajectory in the surface, and a half edge h as trajectory oriented outward from h/s_0 .

Notation 4.24. With the notations of the above observation, denote by $\text{comb}^{\mathbb{R}}_{\mathbf{p}}$ the map between surfaces and open metric ribbon graphs, defined by $(G, \ell) = \text{comb}^{\mathbb{R}}_{\mathbf{p}}(\Sigma)$. Write also $(G, \ell) = \text{comb}^{\mathbb{R}}(\Sigma, \mathbf{p})$.

Definition 4.25. The *normalization* $Norm(G)$ of a stable connected open ribbon graph G is the unique smooth, not necessarily connected, open ribbon graph, defined in the following way. If G is smooth, $Norm(G) = G$. Otherwise the vertex set is $\tilde{V}^I \cup \tilde{V}^B \cup SC^C(G) \cup SC^O(G)$, shrunk components are isolated vertices in the graph, and the half edges are $H^I \cup H^B$. The genus and topological defects of vertices in $\tilde{V}^I \cup \tilde{V}^B$ are 0. For a shrunk component v , $g^{Norm(G)}(v) = g(v)$, $d^{N(v)}(v) = d(v)$. The marking defect is defined by $d^{N(v)}$. In particular $B(v) = (N^B)^{-1}(v) \cup (f^B)^{-1}(v)$.

For any connected component C of $Norm(G)$, not in $SC^C(G) \cup SC^O(G)$, define s_0, s_1, f^I as those induced from G . We define $f^{P_0} : \mathcal{P}_{0C} \rightarrow V^I(C)$ as follows. $\mathcal{P}_{0C} = (\mathcal{P}_{0C} \cap V^I(C)) \cup (\mathcal{P}_{0C} \cap \mathcal{I})$. $\mathcal{P}_{0C} \cap V^I(C)$ is the set of new markings of perimeter 0, which are vertices of \tilde{V}^I that are preimages of nodes. On them f^{P_0} is the inclusion. $\mathcal{P}_{0C} \cap \mathcal{I}$ is the set of $i \in \mathcal{P}_0$ with $f^{P_0}(i) = v$ a marked point of perimeter 0, and $N^{-1}(v) \in C$. Define $(f^C)^I(i) = N^{-1}(v)$. Define $f^B : \mathcal{B}_C \rightarrow V^B(C)$ similarly.

The normalization $Norm(G)$ of a marked graph is the marked graph whose underlying graph is the normalization of the underlying graph of G , new marked points are marked 0.

Write $Norm : Norm(G) \rightarrow G$ to be the evident normalization map.

Note that if v is a shrunk component with at least one edge in G , then $|Norm^{-1}(v)| = |N^{-1}(v)| + 1$.

Observe that the normalization of a trivalent graph is trivalent.

Notation 4.26. There is a canonical injection $B(G) \hookrightarrow B(Norm(G))$. On $B(Norm(G)) \setminus B(G)$ there is a fixed point free involution which we also denote by s_1 , which on preimages of a node which is not a shrunk component it just interchanges its two preimages. If v is a shrunk component, its new boundary markings correspond to elements $u \in (N^B)^{-1}(v)$. Any such u corresponds also to a unique marking w in another *non shrunk* component. Write $s_1 u = w$, $s_1 w = u$.

4.2.3. Moduli of open metric graphs. For a stable open ribbon graph G , denote by $\mathcal{M}_G^{\mathbb{R}}$ the set of all metrics on G , write $\mathcal{M}_G^{\mathbb{R}}(\mathbf{p})$ for the set of all such metrics where the i^{th} face has perimeter $p_i > 0$. Note that $\mathcal{M}_G^{\mathbb{R}} = \mathbb{R}_+^{E(G)} / Aut(G)$. Write $\overline{\mathcal{M}}_G^{\mathbb{R}} = \mathbb{R}_{\geq 0}^{E(G)} / Aut(G)$. We similarly define $\overline{\mathcal{M}}_G^{\mathbb{R}}(\mathbf{p})$.

Construction \ Notation 2. For $e \in E(G)$, the edge between vertices v_1, v_2 , one can define the graph $\partial_e G$, as the graph obtained by contracting e to a point, identifying its vertices to give a new vertex $v_1 v_2$ and

updating the permutations and marking defects as in the closed case. When v_1, v_2 are internal, then so is v_1v_2 . The genus defect is updated as in the closed case, and this determines the whole defect. Suppose v_1 is a boundary vertex. Then so is v_1v_2 . If $v_2 \neq v_1$, then $g(v_1v_2) = g(v_1) + g(v_2)$, if $v_2 \in V^B$, and otherwise $g(v_1v_2) = g(v_1) + 2g(v_2)$. When $v_1 = v_2$, let h_1, h_2 be the half edges of e . Let $\tilde{h}_i \in N^{-1}(v_1)$ be the s_0 -cycle of h_i . Then $g(v_1v_2) = g(v_1) + \delta$, where

$$\delta = \begin{cases} 0, & \text{if } \tilde{h}_1 = \tilde{h}_2, \\ 1, & \text{if } \tilde{h}_1 \neq \tilde{h}_2, \tilde{h}_1, \tilde{h}_2 \in \tilde{V}^B, \\ 2, & \text{otherwise.} \end{cases}$$

$d(v_1v_2) \in D = D(g(v_1v_2), I_{v_1v_2}, B_{v_1v_2})$, or $d(v_1v_2) \in D = D(g(v_1v_2), I_{v_1v_2})$, where $B_{v_1v_2} = (f^B)^{-1}(v_1v_2) \cup (N^B)^{-1}(v_1v_2)$, $I_{v_1v_2} = (f^{P_0})^{-1}(v_1v_2) \cup (N^{P_0})^{-1}(v_1v_2)$. These two sets are already known from what we have constructed so far. In particular, whenever D is trivial, which is always the case for internal vertices, and for boundary vertices it happens when $2g(v_1v_2) + 2|I_{v_1v_2}| + |B_{v_1v_2}| \leq 2$, we know $d(v_1v_2)$. For shortness we will not describe the general update of the topological defect. We do describe a special case of particular importance. Suppose $e \in E^B$, and $v_1 \neq v_2$ are boundary vertices with $d(v_i) \in D(0, \emptyset, B_i)$ where $|B_i| = 2$. This is the case when each v_i is a marked point or a boundary node which is not a shrunk component. Write $B_i = \{\tilde{h}_i, a_i\}$, where \tilde{h}_i is as above. Suppose $h_2 \in H^B$, that is, its orientation disagrees with the orientation of the boundary. Then $d(v_1v_2) \in D(0, \emptyset, \{a, a_1, a_2\})$, where a is the new cycle of s_0h_2 , obtained from concatenating \tilde{h}_1, \tilde{h}_2 after erasing h_1, h_2 , $d(v_1v_2)$ is the element which corresponds to cyclic order $a \rightarrow a_1 \rightarrow a_2$.

Suppose $E' = \{e_1, \dots, e_r\} \subseteq E$, then there is an identification between $E(G) \setminus E'$ and $E(\partial_{e_1, \dots, e_r} G)$. Throughout this paper we shall use this identification without further comment.

If a graph G' is obtained from G by a sequence of edge contractions, we have a canonical map $\mathcal{M}_{G'}^{\mathbb{R}} \hookrightarrow \overline{\mathcal{M}}_G^{\mathbb{R}}$. We say that $\mathcal{M}_{G'}^{\mathbb{R}}$ is a face of $\mathcal{M}_G^{\mathbb{R}}$. Write $\overline{\mathcal{M}}_{g,k,l}^{\mathbb{R} \text{ comb}} = \coprod \overline{\mathcal{M}}_G^{\mathbb{R}} / \sim = \coprod \mathcal{M}_G^{\mathbb{R}}$, where the union is over all open (g, k, l) -ribbon graphs, and \sim is induced by edge contractions. Write $\mathcal{M}_{g,k,l}^{\mathbb{R} \text{ comb}}$ for the locus which is the union over smooth graphs. Define $\overline{\mathcal{M}}_{g,k,l}^{\mathbb{R} \text{ comb}}(\mathbf{p}), \mathcal{M}_{g,k,l}^{\mathbb{R} \text{ comb}}(\mathbf{p})$ by restricting perimeters to be p_i . In all cases we define the cell structure using edge contractions.

The pointwise maps $\text{comb}^{\mathbb{R}}$ induce moduli maps

$$\text{comb}^{\mathbb{R}} : \overline{\mathcal{M}}_{g,k,l}^{\mathbb{R}} \times \mathbb{R}_+^l \rightarrow \overline{\mathcal{M}}_{g,k,l}^{\mathbb{R} \text{ comb}}, \quad \text{comb}^{\mathbb{R}}_{\mathbf{p}} : \overline{\mathcal{M}}_{g,k,l}^{\mathbb{R}} \rightarrow \overline{\mathcal{M}}_{g,k,l}^{\mathbb{R} \text{ comb}}(\mathbf{p}),$$

which sends a stable open surface and a set of perimeters to the corresponding graph.

Lemma 4.27. $\overline{\mathcal{M}}_{g,k,l}^{\mathbb{R} \text{ comb}}$ with the cell structure defined above is a piecewise smooth Hausdorff orbifold with corners. This is the finest topology on the moduli of (g, k, l) -graphs such that the map $\text{comb}^{\mathbb{R}}$ is continuous. $\overline{\mathcal{M}}_{g,k,l}^{\mathbb{R} \text{ comb}}(\mathbf{p})$ is compact for any \mathbf{p} . $\text{comb}^{\mathbb{R}} : \mathcal{M}_{g,k,l}^{\mathbb{R}} \times \mathbb{R}_+^l \simeq \mathcal{M}^{\mathbb{R} \text{ comb}}$. Moreover, the analogous claims remain true if we declare some, but not all, of the internal marked points to have perimeters 0. In fact, for any effective dual graph Γ , the map $\text{comb}^{\mathbb{R}}$ restricted to $\mathcal{M}_{\Gamma}^{\mathbb{R}} \times \mathbb{R}_+^l$ is an isomorphism onto its image.

The proof is similar to the closed case, see [26, 17] for a proof of the analogous theorem.

4.3. JS Stratification for the graded moduli.

4.3.1. *Graded ribbon graphs.* For a metric open ribbon graph, (G, ℓ) , write

$$\tilde{Z}_{G,\ell} = H_0(\text{For}_{\text{spin}}^{-1}((\text{comb}^{\mathbb{R}})^{-1}(G, \ell)), Z_{G,\ell} = H_0(\text{For}_{\text{spin}}^{-1}((\text{comb}^{\mathbb{R}})^{-1}(G, \ell)).$$

For any two generic metrics ℓ, ℓ' the sets $Z_{G,\ell}, Z_{G,\ell'}$ are non canonically isomorphic, see Remark 2.42. For any G , let Z_G be the set $Z_{G,\ell}$ for a fixed generic ℓ . Define \tilde{Z}_G similarly.

Definition 4.28. A *metric spin ribbon graph with a lifting* (G, z, ℓ) is a metric ribbon graph together with $z \in \tilde{Z}_{G,\ell}$. The graph is called *graded* when $z \in Z_{G,\ell}$. A graded graph is a pair $(G, z), z \in Z_G$.

The *normalization* $\text{Norm}(G, z, \ell)$ of (G, z, ℓ) is the smooth, not necessarily connected graph $\coprod (G_i, \ell_i, z_i)$, where (G_i, ℓ_i) are the components of $\text{Norm}(G, \ell)$, and z_i are the classes of twisted spin structures with a lifting, induced by Proposition 2.19. A half node is *legal* if it is legal as a marked point in the graded structure of $\text{Norm}(G, z)$.

It follows from Proposition 4.23 that a graded surface, together with perimeters $\{p_i\}_{i \in \mathcal{I}}$, defines a unique graded metric graph (G, z, ℓ) , where (G, ℓ) is embedded in $K_{\mathcal{B}, \mathcal{P}_0}(\text{For}_{\text{spin}}(\Sigma))$, as in Proposition 4.23 and z is the class of graded spin structures which contains the graded structure of Σ . In particular, for (G, ℓ) generic and effective, $Z_G = Z_{G,\ell}$ is isomorphic to $\text{Spin}(\Sigma)$, and any element z of it correspond to a unique graded structure.

Moreover, by Corollary 2.22 in this case Z_G is in a one to one correspondence with isomorphism classes of tuples $(\mathbb{S}_1, \dots, \mathbb{S}_r)$ where each \mathbb{S}_i is a spin structure with a lifting on the i^{th} component of $\text{Norm}(\Sigma)$,

under the constraints that all original boundary marked points are legal, for any boundary node of Σ exactly one half is legal and every shrunk boundary is a compatible Ramond.

Definition 4.29. A graded spin graph (G, z) , with or without a metric ℓ , is called *effective* if G is effective, and z is a graded spin structure in which for every shrunk component $v \in V(G)$, all boundary marked points of the isolated component in $Norm^{-1}(v)$ are legal. In case v is not isolated, it is equivalent to all half nodes in $(N^B)^{-1}(v)$ being illegal. An efficient graded (G, z) is *trivalent* if G is trivalent. The graph is *smooth* if its underlying graph is.

Denote by \mathcal{SR}^0 the set of isomorphism classes of graded smooth trivalent ribbon graphs, write \mathcal{R}^0 for the set of their underlying open ribbon graphs. Denote by $\mathcal{SR}_{g,k,l}^0 \subseteq \mathcal{SR}^0$ the subset whose faces are marked $[l]$ and the boundary points are marked by $[k]$. Define $\mathcal{R}_{g,k,l}^0$ similarly.

Let $\tilde{\mathcal{SR}}_{g,k,l}^0$ be the collection of all graphs in $\mathcal{SR}_{g,k,l}^0$ with an odd number of boundary marked points on each boundary component. Define $\tilde{\mathcal{R}}_{g,k,l}^0$ similarly.

Note that in a trivalent graph, by definition if v is a shrunk component, the unique ghost component in $Norm^{-1}(v)$ has all its marked points *legal*.

An immediate corollary of Proposition 2.35, which can be taken as an alternative definition of $\mathcal{R}_{g,k,l}^0$, is

Corollary 4.30. $\mathcal{R}_{g,k,l}^0 \neq \emptyset$ exactly when $2|g+k-1$. In this case it is exactly the collection of trivalent smooth graphs.

Notation 4.31. We define the map *comb* between graded surfaces and graded metric ribbon graphs by $comb(\Sigma, \mathbb{S}, s, \mathbf{p}) = (G, z, \ell)$ where $(G, \ell) = comb^{\mathbb{R}}(\Sigma, \mathbf{p})$ and $z \in Z_{G,\ell}$ is the corresponding class. Write $comb_{\mathbf{p}} = comb(-, -, -, \mathbf{p})$. Write $For_{spin}^{comb}(G, z, \ell) = (G, \ell)$.

Proposition 4.32. Suppose $comb(\Sigma, \mathbf{p}) = (G, z, \ell)$.

- (a) Then $comb(Norm(\Sigma), \mathbf{p}) = Norm(G, z, \ell)$, where preimages of nodes in Σ will be internal markings of perimeter 0.
- (b) Suppose Σ' is obtained from Σ by forgetting an illegal marked point v which makes no component become unstable. Write $(G', z', \ell') = comb(\Sigma', \mathbf{p})$. Then $z = z'$ canonically, and (G', ℓ') is obtained from (G, z, ℓ) by the following procedure. If $\deg(v) = 2$, remove it from the graph, unite its two edges e_1, e_2 to one edge

e , define $\ell'(e) = \ell(e_1) + \ell(e_2)$ and for the other edges $\ell' = \ell$. Otherwise the graph and metric do not change, but the marking of v is removed.

Proof. The first item is a consequence of Corollary 4.14. The second follows from Corollary 4.13 and Observation 2.48. \square

4.3.2. Combinatorial moduli for graded surfaces, bundles and forms.

Denote by $\overline{\mathcal{M}}_{g,k,l}^{comb}$ the set of metric graded (g, k, l) -ribbon graphs. Endow it with the finest topology so that *comb* is continuous. Write $\overline{\mathcal{M}}_{g,k,l}^{comb}(\mathbf{p})$ for the subspace of graphs with fixed perimeters \mathbf{p} . Define $\mathcal{M}_{g,k,l}^{comb}$ as the subspace of smooth graphs. Define similarly $\mathcal{M}_{g,k,l}^{comb}(\mathbf{p})$.

The pointwise maps *comb* induce moduli maps

$$comb : \overline{\mathcal{M}}_{g,k,l} \times \mathbb{R}_+^l \rightarrow \overline{\mathcal{M}}_{g,k,l}^{comb}, \quad comb = comb_{\mathbf{p}} : \overline{\mathcal{M}}_{g,k,l} \rightarrow \overline{\mathcal{M}}_{g,k,l}^{comb}(\mathbf{p}),$$

which sends a stable graded surface and a set of perimeters to the corresponding graph.

Lemma 4.33. *Suppose $2|g+k-1$. Then $\overline{\mathcal{M}}_{g,k,l}$, $\overline{\mathcal{M}}_{g,k,l}(\mathbf{p})$ are Hausdorff orbifolds with corners, the latter is compact.*

The map For_{spin}^{comb} is continuous. Moreover, it is an orbifold branched cover, and over any $\mathcal{M}_G^{\mathbb{R}}$ it is an orbifold cover.

The maps $comb, comb_{\mathbf{p}}$ are isomorphisms onto their images when restricted to the open dense subsets $\mathcal{M}_{g,k,l} \times \mathbb{R}_+^l, \mathcal{M}_{g,k,l}$. $comb_{\mathbf{p}}$ induces an orientation on $\overline{\mathcal{M}}_{g,k,l}^{comb}$, with this orientation $\deg(comb_{\mathbf{p}}) = 1$. Analogous claims are true if we declare some, but not all, of the internal marked points to have perimeters 0. In addition, for an effective graded dual graph Γ , the maps $comb, comb_{\mathbf{p}}$ restricted to $\mathcal{M}_{\Gamma} \times \mathbb{R}_+^l, \mathcal{M}_{\Gamma}$ are isomorphisms onto their images.

The proof is similar to the closed case and will be omitted. The orientation on $\overline{\mathcal{M}}_{g,k,l}^{comb}$ will be constructed explicitly later.

We now study the cell structure of $\overline{\mathcal{M}}_{g,k,l}^{comb}$. For a generic $\ell \in \mathcal{M}_G^{\mathbb{R}}$, choose $z \in Z_G = Z_{G,\ell}$, define $\mathcal{M}_{(G,z)}$ to be the connected component of $(For_{spin}^{comb})^{-1}(\mathcal{M}_G^{\mathbb{R}})$ which contains (G, z, ℓ) .

It follows from Lemma 4.33 that $(For_{spin}^{comb})^{-1}(\mathcal{M}_G^{\mathbb{R}})$ is an orbibundle over $\mathcal{M}_G^{\mathbb{R}}$, with a generic fiber Z_G . Since $\mathcal{M}_G^{\mathbb{R}} = \mathbb{R}_+^{E(G)} / Aut(G)$, such a bundle must be of the form

$$(For_{spin}^{comb})^{-1}(\mathcal{M}_G^{\mathbb{R}}) \simeq (\mathbb{R}_+^{E(G)} \times Z_G) / Aut(G),$$

for some action of $Aut(G)$ we now explain.

Let $C \subseteq \mathcal{M}_G^{\mathbb{R}}$ be the locus of generic metrics. Except from some borderline cases which can be treated separately its complement is of

real codimension at least 3. Over C the fiber of the bundle is always of size $|Z_G|$. Denote the bundle by E , and let $\overline{E} \rightarrow \overline{C}$ be the pullback to the preimage of C with respect to the $\text{Aut}(G)$ -quotient. $\pi_1(\overline{C})$ is trivial, as $\mathbb{R}^{E(G)} \setminus \overline{C}$ is of codimension at least 3. Thus \overline{E} must be trivial, and is hence isomorphic to $\overline{C} \times Z_G$.

Let $\overline{\ell} \in \overline{C}$ be any point, let ℓ be its image in C . Recall that as an orbispace, $\text{Aut}(G) \simeq \pi_1(\overline{C}/\text{Aut}(G), \ell)$, and this isomorphism can be made explicit as follows: For $g \in \text{Aut}(G)$, choose any path $\bar{\gamma}^g : [0, 1] \rightarrow \overline{C}$, with $\bar{\gamma}_0^g = \overline{\ell} \in \mathbb{R}_+^{E(G)}$, $\bar{\gamma}_1^g = g \cdot \overline{\ell}$, and set γ_g to be its $\bar{\gamma}^g$ to C .

Parallel transport $z = z_0$ along γ^g to get z_1 . This can be done as the fiber is 0-dimensional. Define $g \cdot (\overline{\ell}, z) = (g \cdot \overline{\ell}, z_1)$. This action is independent of choices, and can be defined continuously over all \overline{E} . This gives us the orbibundle structure over C . Again by continuity, it can be uniquely extended to an action on $\mathbb{R}_+^{E(G)} \times Z_G$.

Note that in particular, we have defined an action of $\text{Aut}(G)$ on Z_G . Define the group $\text{Aut}(G, z)$ as the subgroup of $\text{Aut}(G)$ which leaves z invariant. Then $\mathcal{M}_{(G,z)} = \mathbb{R}_+^{E(G)} / \text{Aut}(G, z)$. Define $\mathcal{M}_{(G,z)}$ as the subsimplex where the perimeters are \mathbf{p} . Write $\overline{\mathcal{M}}_{(G,z)} = \mathbb{R}_{\geq 0}^{E(G)} / \text{Aut}(G, z)$, and define $\overline{\mathcal{M}}_{(G,z)}(\mathbf{p})$ similarly.

Notation 4.34. For $e \in E(G)$, define the *edge contraction* to be $\partial_e(G, z) = (\partial_e G, \partial_e z)$, where $\partial_e z \in Z_{\partial_e G}$ is defined by the cell structure.

An explicit description for the special case of trivalent graphs appears in Subsection 5.1.2.

If a graph (G', z') is obtained from (G, z) by a sequence of edge contractions, we have a canonical map $\mathcal{M}_{(G',z')} \hookrightarrow \overline{\mathcal{M}}_{(G,z)}$. We say that $\mathcal{M}_{(G',z')}$ is a face of $\mathcal{M}_{(G,z)}$. This gives the cell complex structure to $\overline{\mathcal{M}}_{g,k,l}^{comb}$, which agrees with topology, and we can now write $\overline{\mathcal{M}}_{g,k,l}^{comb} = \coprod \overline{\mathcal{M}}_{(G,z)} / \sim = \coprod \mathcal{M}_{(G,z)}$, where the union is over all connected components which correspond to graded (g, k, l) -ribbon graphs, and \sim is induced by edge contractions. Denote the quotient by \sim map by Ξ .

A graph (G, z) corresponds to boundary strata of $\overline{\mathcal{M}}_{g,k,l}^{comb}$, that is $\mathcal{M}_{G,z} \subseteq \text{comb}(\partial \overline{\mathcal{M}}_{g,k,l} \times \mathbb{R}_+^l)$ if and only if it has at least one boundary node. In this case we call it a *boundary graph*.

The combinatorial S^1 -bundles \mathcal{F}_i , $i \in [l]$, are defined as in Definition 4.8. Again these carry a natural piecewise smooth structure, compatible with the natural piecewise smooth structures on $\overline{\mathcal{M}}_{g,k,l}^{comb}$. The forms $\alpha_i, \omega_i, \bar{\alpha}_i, \bar{\omega}_i, \bar{\omega}$ defined as in Definition 4.8 and Equation 19.

Notation 4.35. Any (d, l) -set L is associated a vector bundle $E_L = \sum_{i=1}^d \mathbb{L}_{L(i)} \rightarrow \overline{\mathcal{M}}_{g,k,l}$, and sphere bundles $S(E_L)$ and $S_L = S((\mathcal{F}_{L(i)})_{i=1}^d)$, as in Construction 1. Define an angular form Φ_L for S_L by Formula 15, and using Kontsevich's forms α_i, ω_i , for the copy \mathcal{F}_i , of the $L(i)^{th}$ S^1 -bundle. Note that for $j \in [l]$, different $i_1, i_2 \in L_j$ yield $\omega_{i_1} = \omega_{i_2}$, since they do not contain angular variables. We shall therefore denote ω_i by $\omega_{L(i)}$. On the other hand, it does give rise to different forms, $\alpha_{i_1} \neq \alpha_{i_2}$, since the ϕ variables are different. Still, when it will be clear from context we shall write $\alpha_{L(i)}$ instead of α_i to indicate we are considering a form on $\mathcal{F}_{L(i)}$. Write

$$\omega_L = -d\Phi_L = \bigwedge_{i=1}^d \omega_{L(i)}, p^{2L} = \prod p_{L(i)}^2, \bar{\omega}_L = p^{2L} \omega_L, \bar{\Phi}_L = p^{2L} \Phi_L.$$

When it is not clear from context, we write α^G to indicate the specific graph G . The same remark goes to the other forms.

Exactly as in the closed case, we have

Lemma 4.36. (a) For $i \in [l]$, $\text{comb}^* \mathcal{F}_i \simeq S^1(\mathbb{L}_i)$ canonically. As a result, $\text{comb}^* S_L \simeq S(E_L)$ canonically.
(b) α_i, ω_i are piecewise smooth angular 1-form and Euler 2-form for $S^1(\mathbb{L}_i)$. Φ_L is an angular form of S_L , ω_L its Euler form.
(c) For $(G, z) \in \mathcal{SR}_{g,k,l}^0$, there is a canonical identification $(\mathcal{F}_i \rightarrow \overline{\mathcal{M}}_{(G,z)}) \simeq \Xi^*(\mathcal{F}_i \rightarrow \overline{\mathcal{M}}_{g,k,l}^{\text{comb}})$. Similarly for the bundles S_L .

Notation 4.37. Let (G, z, ℓ) be a metric graded graph. Define the graph $\tilde{\mathcal{B}}(G, z, \ell) = (\tilde{\mathcal{B}}G, \tilde{\mathcal{B}}z, \tilde{\mathcal{B}}\ell)$ by first taking the normalization of (G, z, ℓ) and then forgetting isolated components and the new illegal marked points, as in Proposition 4.32. Let $\tilde{\mathcal{B}} : \mathcal{M}_{(G,z)} \rightarrow \mathcal{M}_{(\tilde{\mathcal{B}}G, \tilde{\mathcal{B}}z)}$ the induced map on the moduli.

Observe that

Observation 4.38. For any graded graph (G, z) , and a face marked i , $\mathcal{F}_i \rightarrow \mathcal{M}_{(G,z)} \simeq \tilde{\mathcal{B}}^*(\mathcal{F}_i \rightarrow \mathcal{M}_{\tilde{\mathcal{B}}(G,z)})$ canonically. A similar claim holds for S_L .

The observation follows from the natural identification of the boundary of the i^{th} faces in $G, \tilde{\mathcal{B}}G$.

Proposition 4.39. A special canonical multisection s of $S(E_L)$ is a pull back of a multisection s' of S_L .

Proof. Take $\mathcal{M}_\Gamma \subseteq \partial \overline{\mathcal{M}}_{g,k,l}$, let i_1, \dots, i_r be labels of internal tails, one for each vertex of Γ . $\text{comb}(\mathcal{M}_\Gamma \times \mathbb{R}_+^l) = \coprod_{(G,z)} \mathcal{M}_{(G,z)}$, where the union is taken over some graded graphs (G, z) . Consider one of them, denote it by (G, z) . Write

$$\Phi_\Gamma = \prod_{j=1}^r \Phi_{\Gamma, i_j}.$$

Consider the diagram

$$(20) \quad \begin{array}{ccc} \text{comb}^{-1} \mathcal{M}_{(G,z)} & \xrightarrow{\Phi_\Gamma} & \text{comb}^{-1} \mathcal{M}_{(\tilde{B}G, \tilde{B}z)} \\ \downarrow \text{comb} & & \downarrow \text{comb} \\ \mathcal{M}_{(G,z)} & \xrightarrow{\tilde{B}} & \mathcal{M}_{(\tilde{B}G, \tilde{B}z)}. \end{array}$$

This diagram commutes, by Proposition 4.32. Now $(\tilde{B}G, \tilde{B}z)$ is smooth, hence the right vertical arrow is an isomorphism, by Lemma 4.33. A special canonical multisection over $\mathcal{M}_\Gamma \times \mathbb{R}_+^l$ is pulled back via Φ_Γ , from $S(E_L) \rightarrow \prod_{j=1}^r \mathcal{M}_{v_i^*(\Gamma)} \times \mathbb{R}_+^l$. Let s be special canonical, we now construct s' with $s = \text{comb}^* s'$. Write $s|_{\text{comb}^{-1} \mathcal{M}_{(G,z)}} = \Phi_\Gamma^*(\text{comb}^*(s''))$ where s'' is a multisection of $S_L \rightarrow \mathcal{M}_{(\tilde{B}G, \tilde{B}z)}$. Define $s'|_{\mathcal{M}_{(G,z)}} = \tilde{B}^* s''$. These multisections for different strata evidently glue. \square

Definition 4.40. A *special canonical multisection* of $S_L \rightarrow \overline{\mathcal{M}}_{g,k,l}^{\text{comb}}$ is a multisection s with $\text{comb}^* s$ being special canonical. A *special canonical multisection* of $S_L \rightarrow \overline{\mathcal{M}}_{(G,z)}$ is a Ξ -pull back of a special canonical multisection on $\overline{\mathcal{M}}_{g,k,l}^{\text{comb}}$. Write $s^{(G,z)}$ for the restriction of s to $\overline{\mathcal{M}}_{(G,z)}$.

The proof of proposition yields the following immediate corollary.

Corollary 4.41. Suppose (G, z) is a boundary (g, k, l) -graded ribbon graph, s is a special canonical multisection of S_L , where L is a (d, l) -set, restricted to the boundary cell $\mathcal{M}_{(G,z)}$ then $s = \tilde{B}^* s'$ where s' is a multisection of $S_L \rightarrow \mathcal{M}_{\tilde{B}(G,z)}$.

The main Result of this section is that the descendents can be calculated over the combinatorial moduli.

Lemma 4.42. Let s be a special canonical multisection for $S(E_L)$. Denote by s' the multisection on S_L with $s = \text{comb}^* s'$. Then

$$(21) \quad \int_{\overline{\mathcal{M}}_{g,k,l}} e(S(E_L), s) = \int_{\overline{\mathcal{M}}_{g,k,l}^{\text{comb}}} e(S_L, s').$$

The orientations are the ones induced on the combinatorial moduli by comb_* .

The proof is an immediate consequence of Lemmas 4.33, 4.36, by functoriality of the relative Euler class.

4.3.3. Intersection numbers as integrals over the combinatorial moduli. We can now use the natural piecewise linear structure on $\overline{\mathcal{M}}_{g,k,l}^{comb}$ and the associated bundles to write an explicit integral formula for them.

Definition 4.43. A *bridge* in a graded graph (G, z) is either a boundary edge between two distinct special legal boundary points or an internal edge between two boundary vertices. Denote by $Br(G, z)$ the set of bridges of (G, z) . Usually we shall omit z from the notation and write $Br(G)$ instead. A *compatible sequence of bridges* $\{e_1, \dots, e_r\}$ is a sequence of bridges such that e_{i+1} is a bridge in $\partial_{e_1, \dots, e_i} G$ for all i .

Suppose e is a bridge and $h \in H^I$ satisfies $h/s_1 = e$. Set $h' = s_2 h$. We define $\partial_e h \in HN(\partial_e G)$ to be the unique vertex $v \in V(Norm(\partial_e G))$ with $h'/s_0 = v$, where we consider h' as an edge of $Norm(\partial_e G)$, using the canonical identification. When there is $h \in H^B$ with $h/s_1 = e$, contracting e creates a shrunk component v , which is identified with a ghost component of $Norm(G)$, see Figure 3, *d*. We denote by $\partial_e h \in B(v)$ the marking which is the s_0 -cycle of $s_2(s_1 h)$ in $(N^B)^{-1}(v)$. This is equivalent to writing $\partial_e h = s_1 \partial_e(s_1 h)$, recalling Notation 4.26.

The following observation is immediate

Observation 4.44. (a) $\dim \mathcal{M}_{(G,z)}(\mathbf{p}) = \dim \overline{\mathcal{M}}_{g,k,l}$ if and only if $(G, z) \in \mathcal{SR}_{g,k,l}^0$.
 (b) In addition, (G, z) is a boundary graph if and only if it can be represented as $\partial_{e_1, \dots, e_r}(G', z')$, where $(G', z') \in \mathcal{SR}_{g,k,l}^0$, and at least one e_i is a bridge. The only boundary graphs whose (G, z) whose moduli is of full dimension $\dim \overline{\mathcal{M}}_{g,k,l} - 1$, are those which can be written as $\partial_e(G', z')$, for $(G', z') \in \mathcal{SR}_{g,k,l}^0$, $e \in Br(G')$.
 (c) If $\{e_1, \dots, e_r\}$ is a compatible sequence of bridges in a trivalent graph (G, z) then $\partial_{e_1, \dots, e_r}(G, z)$ is trivalent. Any trivalent can be written in a unique way as $\partial_{e_1, \dots, e_r}(G, z)$, where (G, z) is smooth trivalent and $\{e_1, \dots, e_r\}$ is a compatible sequence of bridges.

See Figure 3, *d, e* for examples.

Using Observation 4.44, Lemma 4.42 and Proposition 3.3 we immediately get

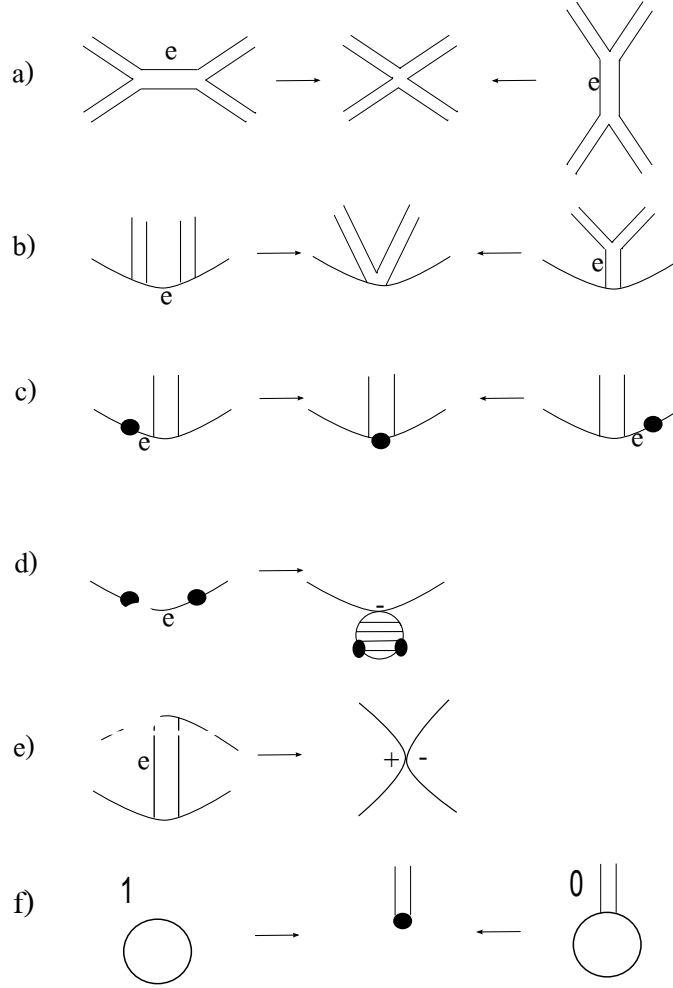


FIGURE 3. Edge contractions and Feynman moves.

Lemma 4.45. *Let L be a (d, l) – set where $d = \frac{3g-3+k+2l}{2}$, and let s be a special canonical multisection for S_L . Then*

$$(22) \quad 2^{\frac{g+k-1}{2}} \langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \rangle =$$

$$(23) \quad \sum_{(G,z) \in \mathcal{SR}_{g,k,l}^0} \int_{\mathcal{M}_{(G,z)}(\mathbf{p})} \omega_L + \sum_{(G,z) \in \mathcal{SR}_{g,k,l}^0, [e] \in [Br(G)]} \int_{\overline{\mathcal{M}}_{\partial_e(G,z)}(\mathbf{p})} s^* \Phi_L.$$

The orientations are the ones induced on the combinatorial moduli by $comb_$.*

Remark 4.46. The formalism of piecewise linear forms and their integration is treated, for instance, in [26].

Construction\Notation 3. For later purposes we now define Feynman moves in edges. Suppose that G is a trivalent graph, $e \in E \setminus \text{Br}(G)$. If e is a boundary edge, we require that least one of its vertices is not a special point. Define the graph $G_e = G$, in case e is a loop. Otherwise, define G_e as the graph obtained from G by first contracting e and then reopening it in the unique different possible way, see Figure 3, a, b, c, f.

Proposition 4.47. For any graded trivalent (G, z) and $e \in E(G) \setminus \text{Br}(G)$, there is a unique graded structure z_e such that if G is smooth, $\mathcal{M}_{(G_e, z_e)}$ is the unique codimension 0 cell of $\overline{\mathcal{M}}_{g, k, l}^{\text{comb}}$ adjacent to $\mathcal{M}_{(G, z)}$ along $\mathcal{M}_{\partial_e(G, z)}$. Otherwise, write $(G, z) = \partial_{e_1, \dots, e_r}(H, w)$, $e_1, \dots, e_r \in E(H)$, with (H, w) trivalent and smooth. Then

$$(G_e, z_e) = \partial_{e_1, \dots, e_r}(H_e, w_e).$$

Whenever e is a loop, $z_e \in Z_G$ is the graded structure which is identical to z except an opposite lifting on the boundary component e .

Proof. Indeed, for a smooth trivalent G , $\partial_e \mathcal{M}_{G, z}$ is a codimension 1 face which is not a boundary, hence must be adjacent to a single cell codimension 0 cell. Since $\text{For}_{\text{spin}}^{\text{comb}}$ is continuous, this cell must be of the form $\mathcal{M}_{(G, z_e)}$ or $\mathcal{M}_{(G_e, z_e)}$. In case e is a loop, $\partial_e \mathcal{M}_{(G, z)}^{\mathbb{R}}$ is a boundary of the moduli which corresponds to strata with shrunk boundary. There only the lifting in that boundary changes by Remark 2.43. Since the covering map $\mathcal{M}_{g, k, l}^{\text{comb}} \simeq \mathcal{M}_{g, k, l} \rightarrow \mathcal{M}_{g, k, l}^{\mathbb{R}} \simeq \mathcal{M}_{g, k, l}^{\mathbb{R}, \text{comb}}$ is a honest covering, when e is neither a loop nor a bridge, the neighboring cell must be $\mathcal{M}_{(G_e, z_e)}$. The rest of the claim follows from the cell structure and Observation 4.44, part (c). \square

The operations $G \rightarrow G_e$, $(G, z) \rightarrow (G_e, z_e)$ are called Feynman moves.

5. TRIVALENT AND CRITICAL NODAL GRAPHS

It follows from Lemma 4.45 that all intersection numbers can be calculated as integrals over the highest dimensional cells of $\overline{\mathcal{M}}_{g, k, l}^{\text{comb}}$, and of $\partial \overline{\mathcal{M}}_{g, k, l}^{\text{comb}}$. These cells are parameterized by graphs with some extra structure. In this section we shall find a combinatorial interpretation to the extra structure, use it to describe the boundary conditions and write an explicit expression for the canonical orientations.

Definition 5.1. Let G be any open ribbon graph. A *good ordering* is a bijection $n : H^I \rightarrow |H^I|$, which satisfies the following properties. First, if $i(h) < i(h')$, that is h belongs to face marked i , and h' to face marked $i' > i$, then $n(e) < n(e')$. Thus, half edges of the same face

are clustered together. Second, the ordering n , when restricted to half edges of a single face, agrees with the counterclockwise ordering.

5.1. Kasteleyn orientations. Fix a graph $G \in \mathcal{R}_{g,k,l}^0$, from now till the end of this subsection.

Definition 5.2. Consider the set \mathcal{A} of all assignments $H^I \rightarrow \mathbb{Z}_2$. A *vertex flip* is the involution $f_v : \mathcal{A} \rightarrow \mathcal{A}$ defined as follows. For $A \in \mathcal{A}$, $f_v A$ is the assignment which satisfies the following condition. $f_v A(h) \neq A(h)$ if and only if exactly one of the vertices of h , $h/s_0, s_1(h)/s_0$ is v .

A *Kasteleyn orientation on G* is an assignment $K \in \mathcal{A}$ which satisfies the following conditions.

- (a) If h belongs to a boundary edge, that is $s_1(h) \in H^B$, then

$$K(h) = 1.$$

- (b) For other half edge h

$$K(h) + K(s_1(h)) = 1.$$

- (c) For every face i ,

$$\sum_{h \in H_i} K(h) = 1.$$

For convenience extend K to H^B by 0, so that Property (b) holds for any half edge. $K(G)$ will stand for the set of all Kasteleyn orientations of G . Vertex flips act on the set $K(G)$. Two Assignments or two Kasteleyn orientations are *equivalent* if they differ by vertex flips. Write $[K(G)]$ for the set of equivalence classes of Kasteleyn orientations, and $[K]$ for the equivalence class of K .

Observation 5.3. Equivalent assignments give the same value to any half edge of a bridge.

Definition 5.4. The *legal side* of a bridge e is the half edge $h \in s_1^{-1}(e)$ with $K(h) = 0$. The other side is *illegal*.

The main goal of this subsection is to show that there is a natural bijection between $\mathcal{SR}_{g,k,l}^0$ and $\{(G, [K]) | G \in \mathcal{R}_{g,k,l}^0, [K] \in [K(G)]/Aut(G)\}$.

We first show how a graded structure induces an element in $[K(G)]$. Take a graded surface (Σ, \mathbb{S}, s) whose corresponding embedded ribbon graph, defined by the JS differential, is G .

Definition 5.5. Let $v \in V^I$, and $\{h_i\}_{i=1,2,3}$, are its three half edges, ordered so that $s_0 h_i = h_{i+1}$. A *choice of lifting for v* is a choice of lifts, $l_{h_i} \in \mathbb{S}_v$ for the oriented $T_v^1 h_i$, for which

$$l_{h_{i+1}} = R_{\theta_i + 2\pi} l_{h_i}, \quad i = 1, 2, 3,$$

where $\theta_i = \angle(T_v h_i, T_v h_{i+1})$.

Let $\partial\Sigma_b$ be a boundary component. Write $H_b = \{h_i\}_{i=1}^m$, where $h_i \in H^I$, are the half edges which are embedded in $\partial\Sigma_b$, ordered so that $h_{i+1} = s_1(s_2^{-1}(s_1(h_i)))$. Put $v_i = h_i/s_0$. A *lifting* for $\partial\Sigma_b$ is the unique choice of lifts $l_h \in \mathbb{S}_{v_i}$ of $T_{v_i}^1 h$, for any i and any $h \in H_{v_i}$, which satisfies the following requirements.

- (a) For $h \in s_1 H_b$, $l_h = s(v_i)$.
- (b) If v_i is not a marked point, let $f = s_0 h_i$, and put $\theta = \angle(h_i, f)$. Then $l_f = R_{\theta+2\pi} l_{h_i}$, and $l_{s_0^{-1} h_i} = R_\pi l_{h_i}$.
- (c) If v_i is a marked point $l_{s_0^{-1} h_i} = R_{3\pi} l_{h_i}$.

A *choice of a lifting* is a choice of lifting for any vertex and boundary of the graph.

Note that a choice of a lifting for a boundary does not depend on choices. For an internal vertex, it does not depend on the choice of which half edge is taken to be h_1 , since iterating three times we see that $l_i = R_{8\pi} l_i$, which is true.

The next observation is a consequence of the definition of the graded boundary conditions.

Observation 5.6. Consider a lifting for the boundary $\partial\Sigma_b$. With the above notations, if v_i is a marked point, then $l_{h_i} = R_{2\pi} P(h_{i-1}) l_{h_{i-1}}$. If v_i is a boundary vertex which is not a marked point, then $l_{h_i} = P(h_{i-1}) l_{h_{i-1}}$. In both cases $R_\pi P(h_{i-1}) l_{h_{i-1}} = l_{s_1(h_{i-1})} = l_{s_0^{-1} h_i}$.

Remark 5.7. Iterating Observation 5.6 over all boundary vertices, we are led to the single constraint $l_{h_i} = R_{2k_b\pi} l_{h_i}$, where k_b is the number of boundary marked points of the boundary component $\partial\Sigma_b$. By unwinding the alternations in boundary marked points, we see that $q(\gamma) = k_b + 1$, for γ a simple closed trajectory isotopic to $\partial\Sigma_b$.

A choice of a lifting induces an assignment $K \in \mathcal{A}$ as follows. $K(h) = 1$, if $s_1 h \in H^B$. For an internal half edge h , considered as a trajectory from u to v , we have lifts $l_h, l_{s_1(h)}$ of $T_u^1 h, T_v^1 s_1 h$ respectively. Now, $R_\pi P(h) l_h$ also covers $T_v^1 s_1 h$, hence it equals either $l_{s_1(h)}$ or $R_{2\pi} l_{s_1(h)}$. In the first case we define $K(h) = 1$, otherwise $K(h) = 0$. Write $K(\Sigma, \mathbb{S}, s)$ for the set of all assignments of G induced by choices of liftings.

Definition 5.8. A *vertex lift flip* in a vertex $v \in V^I$ is the involution of the set of choices of lifts, which takes one choice to the choice which differs exactly in the lift at v .

We have the following lemma

Lemma 5.9. *If C, C' are two choices of lifts which differ by a vertex lift flip in v , the corresponding assignments K, K' differ by a vertex flip f_v . The vertex flips act commutatively simply transitively on $K(\Sigma, \mathbb{S}, s)$. The correspondence between choices of lifts and $K(\Sigma, \mathbb{S}, s)$ is a bijection. As a conclusion $|K(\Sigma, \mathbb{S}, s)| = 2^{V^I(G)}$.*

Proof. The first assertion, the commutativity and transitivity of the action are straightforward. The rest will from proving that the action is simple. In order to show this, note that we can think of $K(\Sigma, \mathbb{S}, s)$ as subset of $\mathbb{Z}_2^{H^I}$. This is a vector space and a vertex flip f_v is just an addition of an element $\tilde{f}_v \in \mathbb{Z}_2^{H^I}$, which is s_1 -invariant, and zero everywhere except for edges with exactly one of their ends is v . Thus, we can also think of \tilde{f}_v as an element of \mathbb{Z}_2^E , which vanishes identically on boundary edges. In other words, \tilde{f}_v is canonically a 1-cochain relative to boundary. If ∂ is the coboundary operator on the relative cochain complex defined on Σ by the 1-skeleton G , then $\tilde{f}_v = \partial e_v$, where e_v is the 0-cochain which is 1 only at v . If the action of vertex flips were not simple, there would have been a subset $A \subseteq V^I$ such that

$$\sum_{v \in A} \tilde{f}_v = 0,$$

or equivalently

$$\partial \sum_{v \in A} e_v = 0,$$

so $\sum_{v \in A} e_v = 0$ is closed in $H^0(\Sigma, \partial\Sigma) \simeq H_2(\Sigma)^*$, by Poincaré-Lefschetz duality. But $H_2(\Sigma) = 0$, which means $A = \emptyset$. \square

We now study $K(\Sigma, \mathbb{S}, s)$ more carefully.

Proposition 5.10. *Fix $K \in K(\Sigma, \mathbb{S}, s)$ and $h \in H^I$. Put $v = h/s_0, u = (s_1 h)/s_0$, $f = s_0^{-1} s_1 h$, and, in case u is not a marked point, $f' = s_0^{-2} s_1 h$. Write $\theta = \angle(P(h)T_v^1 h, T_u^1 f) \in (-\pi, \pi)$ and $\alpha = \angle(f', f) \in (0, 2\pi)$, if u is not a marked point. Finally, let $\varepsilon = K(h)$. If l_h, l_f , and when u is not a marked point, also $l_{f'}$, denote the lifts of $T_v^1 h, T_u^1 f, T_u^1 f'$ respectively, induced by K , then we have the following equalities.*

- (a) $l_f = R_{2\pi\varepsilon+\theta} P(h) l_h$.
- (b) $l_{f'} = R_{2\pi(1+\varepsilon)+\theta-\alpha} P(h) l_h$, and $\theta - \alpha \in (-\pi, \pi)$.

For $h \in H^B$, from v to u , write $f = s_2 h$. Put $f' = s_0 s_1 h$ whenever u is not a marked point, and $\theta = \angle(P(h)T_v^1 h, T_u^1 f') \in (-\pi, 0)$. Then, if u is a marked point $R_{2\pi} P(h) l_h = l_f$. Otherwise $P(h) l_h = l_f$, $R_{\theta+2\pi} l_h = l_{f'}$.

Proof. We prove for $h \in H^I$. The proof for boundary half edges is similar and follows from Observation 5.6.

$$\begin{aligned} K(h) = \varepsilon &\Leftrightarrow R_\pi P(h)l_h = R_{(1+\varepsilon)2\pi}l_{s_1(h)} \\ &\Leftrightarrow R_\pi P(h)l_h = R_{(1+\varepsilon)2\pi}(R_{2\pi+\pi-\theta}l_f) \\ &\Leftrightarrow R_\theta P(h)l_h = R_{\varepsilon 2\pi}l_f, \end{aligned}$$

where the equivalence in the second line follows from the definition of a choice of lift in a vertex, while the equivalence with the last line is a consequence of Remark 2.28. The second claim follows from $l_{f'} = R_{-2\pi-\alpha}l_f$ and the cyclic order of the half-edges. \square

We now prove

Lemma 5.11. *If $K \in K(\Sigma, \mathbb{S}, s)$, then K is a Kasteleyn orientation.*

Proof. Property (a) of Kasteleyn orientations is just Observation 5.6. Property (b) is reduced, thanks to Remark 2.28 and the construction of K , to

$$R_\pi P(s_1(h))R_\pi P(h) = R_{2\pi},$$

but this follows from Proposition 2.31 applied to the piecewise smooth curve closed $h \rightarrow \bar{h} \rightarrow h$, where \bar{h} is h with the opposite orientation.

For property (c), let h_1, \dots, h_m be an ordering of H_i such that $s_2(h_j) = h_{j+1}$. Set $v_j = h_j/s_0$. Let l_{h_j} be the lift of $T_{v_j}^1 h_j$ determined by K , using Lemma 5.9. Proposition 2.31 applied to the piecewise smooth curve $\gamma_i = h_1 \rightarrow h_2 \rightarrow \dots \rightarrow h_m \rightarrow h_1$, is equivalent to $P(\gamma_i)l_{h_1} = R_{2\pi}l_{h_1}$. Put $\theta_{j+1} = \angle(P(h_j)T_{v_j}^1 h_j, T_{v_{j+1}}^1 h_{j+1}) \in (-\pi, \pi)$. Now, by Proposition 5.10,

$$R_{\theta_{j+1}}P(h_j)l_{h_j} = R_{\varepsilon_j 2\pi}l_{h_{j+1}}, \quad \varepsilon_j \in \mathbb{Z}_2,$$

where $\varepsilon_j = K(h_j)$. Iterating this equation, for $j = 1, \dots, m$, we get

$$\begin{aligned} l_{h_1} &= R_{2\pi\varepsilon_m+\theta_1}P(h_m)R_{2\pi\varepsilon_{m-1}+\theta_m}P(h_{m-1}) \dots R_{2\pi\varepsilon_1+\theta_2}P(h_1)l_{h_1} = \\ &= R_{2\pi\sum_{i=1}^m \varepsilon_i}R_{\theta_1}P(h_m)R_{\theta_m}P(h_{m-1}) \dots R_{\theta_2}P(h_1)l_{h_1}. \end{aligned}$$

On the other hand, $R_{\theta_1}P(h_m)R_{\theta_m}P(h_{m-1}) \dots R_{\theta_2}P(h_1) = R_{2\pi(1+q(\gamma))}$, by the definition of q . But $q(\gamma) = 0$, so $\sum_{i=1}^m \varepsilon_i$ must be odd. \square

Theorem 5.12. *Let G, Σ be as above. There is a bijection between graded spin structures on Σ and $[K(G)]$.*

Proof. Given a graded spin structure on Σ , we have constructed an equivalence class of Kasteleyn orientations, so that we get a map

$$[K] : \text{Spin}(\Sigma) \rightarrow [K(G)].$$

We shall construct a map Spin in the other direction.

Fix $K \in K(G)$. We first construct the restriction of the spin bundle to G , the 1-skeleton of Σ . For any vertex v , write

$$N_v = \cup_i \{h'_i\},$$

where h'_i are the half open half edges emanating from v , after removing their second endpoint. We define $Spin(K)|_{N_v}$ as the trivial spin cover of $T^1\Sigma|_{N_v}$. On any fiber of $Spin(K)$ there is an action of $\mathbb{R}/4\pi\mathbb{R}$, denote it by R_θ .

For a vertex v , choose sections $l_{h_i} : h'_i \rightarrow Spin(K)|_{h'_i}$, which cover $T_v^1 h_i$, such that for any $h_i \notin H^B$,

$$R_{2\pi+\theta_i}(v)l_{h_i}(v) = l_{s_0(h_i)}(v),$$

where $\theta_i = \angle(T_v^1 h_i, T_v^1 s_0(h_i))$.

The transition map $g_{e', s_1(e)'} : Spin(K)|_{e'} \rightarrow Spin(K)|_{s_1(e)'}$ is given by identifying $R_{2K(e)\pi-\pi}l_h$ and $l_{s_1 h}$, and extending using the $\mathbb{R}/4\pi$ action.

It follows from construction, and from Property (c) of Kasteleyn orientations that for each $i \in [l]$, the spin structure on the boundary of face i of G , which is a topological disk, satisfies Proposition 2.31, and thus can be extended uniquely to the face. Thus, we have constructed a spin structure on Σ . The section $\{l_h\}_{h \in s_1 H^B}$, is evidently a grading. Call this graded spin structure $Spin(K)$. It can be verified easily that equivalent Kasteleyn orientations give rise to the same graded spin structure, and that the maps $[K], Spin$ are inverse to each other. \square

Now that we know that the data of an equivalence class of Kasteleyn orientations is equivalent to the data of a graded spin structure, we may try to calculate q, Q using K .

Definition 5.13. Let $\gamma = (h_1 \rightarrow \dots \rightarrow h_m (\rightarrow h_1))$ be an open (closed) directed path in $G \in \mathcal{R}_{g,k,l}^0$ without backtracking, that is, the directed edge $s_1 h$ cannot follow h in the path. Put $v_i = h_i/s_0$. We say that γ makes a *bad turn* at v_i if $h_{i-1} \in H^I$ and $h_i = s_2 h_{i-1}$, or $h_{i-1} \in H^B$ and $h_i = s_0 s_1 h_{i-1}$ ($i+1$ is taken modulo m in the closed case). Otherwise it makes a *good turn*. $BT(\gamma)$ is the number of bad turns.

Proposition 5.14. Fix $[K]$. With the conventions of the previous definition,

- (a) For γ closed, $q(\gamma) = q_K(\gamma) := 1 + \sum_i K(h_i) + BT(\gamma)$, for any $K \in [K]$.
- (b) For γ open, with $h_1, h_m \in s_1 H^B$, let $\tilde{\gamma}$ be the trajectory obtained from γ after removing small neighborhoods of its endpoints, then $Q(\tilde{\gamma}) = Q_K(\gamma) := 1 + \sum_i K(h_i) + BT(\gamma)$, for any $K \in [K]$.

We defined $\tilde{\gamma}$ in order to avoid marked points as endpoints.

Proof. We prove for closed γ , the proof for open is similar. Fix $K \in [K]$. Recall the correspondence between Kasteleyn orientations and lifts, 5.9, and take the corresponding lift l . Put $\theta_{j+1} = \angle(P(h_j)T^1h_j, T^1h_{j+1}) \in (-\pi, \pi)$. By Proposition 5.10,

$$R_{\theta_{j+1}}P(h_j)l_{h_j} = R_{(\varepsilon_j + bt_{j+1})2\pi}l_{h_{j+1}},$$

where $\varepsilon_j = K(h_j)$, and $bt_{j+1} \in \mathbb{Z}_2$ is 1 if and only if γ makes a bad turn in v_{j+1} . Iterating this equation, for $j = 1, \dots, m$, we get that

$$\begin{aligned} l_{h_1} &= R_{2\pi(\varepsilon_m + bt_1) + \theta_1}P(h_m)R_{2\pi(\varepsilon_{m-1} + bt_m) + \theta_m}P(h_{m-1}) \cdots \\ &\quad \cdots R_{2\pi(\varepsilon_1 + bt_2) + \theta_2}P(h_1)l_{h_1} \\ &= R_{2\pi \sum_{i=1}^m \varepsilon_i + bt_i}R_{\theta_1}P(h_m)R_{\theta_m}P(h_{m-1}) \cdots R_{\theta_2}P(h_1)l_{h_1} \\ &= R_{2\pi(BT(\gamma) + \sum_{i=1}^m \varepsilon_i)}R_{(1+q(\gamma))2\pi}l_{h_1} = R_{2\pi(q(\gamma) + A)}l_{h_1}, \end{aligned}$$

where $A = 1 + BT(\gamma) + \sum_{i=1}^m \varepsilon_i$. And the result follows. \square

Remark 5.15. The first case of the proposition appeared before in [6]. Note that although the formula depends on the orientation of γ , the result is orientation independent in the closed case. Indeed, flipping the orientation changes each $K(h)$ to $K(s_1h) = K(h) + 1$, and interchanges the sets of good turns and of bad turns. Thus, the total change is the number of edges plus the number of vertices of γ , that is, a change by $2m = 0$. The same argument shows that in the open case the result changes by 1 when the orientation is flipped.

Definition 5.16. An automorphism $\phi : G \rightarrow G$ defines an action, ϕ_* on $K(G)$, $[K(G)]$ by

$$(\phi_*K)(h) = K(\phi^{-1}(h)).$$

An automorphism ϕ of $(G, [K])$ is an automorphism ϕ of G for which $\phi_*[K] = [K]$. We write $\text{Aut}(G, [K])$ for the group of these automorphisms.

Proposition 5.17. For any $G \in \mathcal{SR}_{g,k,l}^0$, the map

$$\coprod_{z \in Z_G / \text{Aut}(G)} \mathcal{M}_{(G,z)} \rightarrow \coprod_{[K] \in [K(G)] / \text{Aut}(G)} \mathbb{R}_+^{E(G)} / \text{Aut}(G, [K]),$$

which takes a metric graded graph (G, z, ℓ) to $([K], \ell)$, where $[K]$ is the Kasteleyn orientation associated to the graded spin structure of $\text{comb}^{-1}(G, z, \ell)$ is a homeomorphism.

Proof. It is enough to show that along a path $(\Sigma_t)_{0 \leq t \leq 1}$ in $\text{comb}^{-1}(\mathcal{M}_{(G,z)})$, the equivalence classes $[K_t] = [K_t(\Sigma_t, \mathbb{S}_t, s_t)] \in [K(G)]$ are the same. Take $K_0 \in [K(\Sigma_0, \mathbb{S}_0, s_0)]$. This determines the maps Q_0, q_0 by Proposition 5.14, and the fact that any piecewise smooth path may be isotoped

to a non backtracking one on the 1-skeleton $G \hookrightarrow \Sigma_0$. Now, varying $(\Sigma_t, \mathbb{S}_t, s_t)$ is equivalent to varying the metric ℓ_t on G , in the component $\mathcal{M}_{(G,z)}$ continuously. But then it is evident that the maps Q_t, q_t determined by K_0 on the paths in resulting embedded graph do not change. By Lemma 2.44 we see that $[K_t] = [K_0]$. \square

In light of Proposition 5.17, we can redefine \mathcal{SR}^0 and related moduli.

Notation 5.18. From now on we write

$$\mathcal{SR}_{g,k,l}^0 = \{(G, [K]) | G \in \mathcal{R}_{g,k,l}^0, [K] \in [K(G)]/Aut(G)\}.$$

Define $\mathcal{M}_{(G,[K])} = \mathbb{R}_+^{E(G)}/Aut(G, [K])$, the moduli of metrics on G , together with a fixed equivalence class of Kasteleyn orientations. Put $\overline{\mathcal{M}}_{(G,[K])} = \mathbb{R}_{\geq 0}^{E(G)}/Aut(G, [K])$. Define analogously $\overline{\mathcal{M}}_{(G,[K])}(\mathbf{p})$, and $\overline{\mathcal{M}}_{(G,[K])}(\mathbf{p})$.

Example 5.19. Fix a connected component C of $\overline{\mathcal{M}}_{g,k,l}^{\mathbb{R}}$ with an odd number k_j of boundary marked points on boundary component j . Suppose that smooth surfaces in C have b boundary components and write $g_s = \frac{g-b+1}{2}$. One ribbon graph which corresponds to surfaces in C is the following graph $G \in \mathcal{R}_{g,k,l}^0$, see also Figure 4

$$\begin{aligned} V = & \{v_{j,j+1}^-\}_{j=2,\dots,b} \cup \{v_{j,j+1}^+\}_{j \in [b-1]} \cup \{p_{ji}\}_{j \in [b], i \in [k_j]} \\ & \cup \{v_i^\pm\}_{i=2,\dots,l} \cup \{u_i^\pm, w_i^\pm\}_{i \in [g_s]}. \end{aligned}$$

Only v_i^- are internal vertices, the vertices $p_{ji}, v_{j,j+1}^+, v_{j-1,j}^-$ belong to the j^{th} boundary component. The other boundary vertices belong to the first boundary.

$$H^I = \bigcup_{i \in [b]} H_{bdry,i} \cup H_{bridges} \cup H_{genus} \cup H_{internalmarked},$$

where

- (a) For $j \neq 1$, $H_{bdry,j} = \{e_{ji}\}_{0 \leq i \leq k_j + (1-\delta_{jb})}$ are the boundary edges of boundary component j and of face 1. $e_{ji}/s_0 = p_{j(i-1)}$ for $1 \leq i \leq k_j$. In addition, $e_{j0}/s_0 = v_{j(j+1)}^+$, $(s_1 e_{j0})/s_0 = p_{j0}$. For $j \neq b$, e_{jk_j} connects p_{jk_j} to $v_{(j-1)j}^-$, and $e_{jk_j+1}/s_0 = v_{j-1,j}^-$, $s_1(e_{jk_j+1})/s_0 = v_{j,j+1}^+$. For $j = b$, $e_{bk_b}/s_0 = v_{b-1b}^-$. They are ordered so that $e_{ji+1} = s'_2 e_{ji}$, where $s'_2(e) := s_1(s_2^{-1}(s_1(e)))$, for $e \in s_1 H^B$.
- (b) $H_{bdry,1} = a_1, b_1, c_1, d_1, a_2, \dots, d_{g_s}, h_2, \dots, h_l, e_{10}, e_{11}, \dots, e_{1k_1}$ are the boundary edges of the first boundary, which all belong to face 1, ordered by s'_2 order. $a_1/s_0 = v_{1,2}^+$. $a_i/s_0 = w_{i-1}^-$, for $i > 1$, while $b_i/s_0 = u_i^+$, $c_1/s_0 = w_1^+$, $d_1/s_0 = u_1^-$. Next, $h_2/s_0 =$

- $w_i^+, h_i/s_0 = v_{i-1}^+$, for $i > 1$. Finally $e_{10}/s_0 = v_l^+$, and for $i > 0$, $e_{1i}/s_0 = p_{1i}$.
- (c) $H_{bridges} = \{b_{j,j+1}, \bar{b}_{j,j+1}\}_{j \in [b-1]}$ is the set of bridges between consecutive boundaries.
- $$b_{j,j+1}/s_0 = v_{j,j+1}^+, \bar{b}_{j,j+1} = s_1 b_{j,j+1}, \bar{b}_{j,j+1}/s_0 = v_{j,j+1}^-.$$
- (d) $H_{genus} = \{f_i, \bar{f}_i, g_i, \bar{g}_i\}_{i \in g_s}$ is a set of internal half edges of face 1, such that f_i goes from u_i^+ to u_i^- , $\bar{f}_i = s_1 f_i$, and g_i goes from w_i^+ to w_i^- , $\bar{g}_i = s_1 g_i$.
- (e) $H_{internalmarked} = \{x_i, \bar{x}_i, y_i, \bar{y}_i\}_{i=2,\dots,l}$, is the following set. y_i is the unique edge of face i , $y_i/s_0 = v_i^-$, and $\bar{y}_i = s_1 y_i$. The third half edge of v_i^- is x_i , and $\bar{x}_i = s_1 x_i$, $\bar{x}_i/s_0 = v_i^+$.

We now describe $K(G)$. First of all, $K(h) = 1$ if $s_1 h \in H^B$ or $h = y_i$. There is no constraint on $K(x_i)$, but different values are equivalent by flips in v_i^- . Since there are no more internal vertices, for all other edges there are no constraints and no relations. Thus there is a total number of $2^{2g_s+b-1} = 2^g$ different graded spin structures in this case. Since this is a topological invariant, for any generic open genus g surface which satisfies 5 there are 2^g graded structures.

Remark 5.20. In [21] a notion of parity, or Arf invariant is defined for smooth graded surfaces with an odd number of boundary point for each component. It is defined as follows. Given such a graded surface (Σ, \mathbb{S}, s) , choose a symplectic basis $\{\alpha_i, \beta_i\}_{i \in [g_s]}$ to $H_1(\Sigma)/H_0(\partial\Sigma)$. The quadratic form q factors through this quotient. Define $Arf(\Sigma) = \sum q(\alpha_i)q(\beta_i) \pmod{2}$. This is an isotopy invariant. A spin structure is said to be even if the Arf is 0, otherwise it is odd.

For example, in Example 5.19 a possible choice for the symplectic basis is

$$\alpha_i = b_i \rightarrow c_i \rightarrow \bar{f}_i \rightarrow b_i, \quad \beta_i = c_i \rightarrow d_i \rightarrow \bar{g}_i \rightarrow c_i.$$

Now,

$$q(\alpha_i) = 1 + K(b_i) + K(c_i) + K(\bar{f}_i) + BT(\alpha_i) = K(\bar{f}_i),$$

since there is one bad turn. Similarly, $q(\beta_i) = K(\bar{g}_i)$. Therefore,

$$Arf(\Sigma) = \sum_{i \in [g_s]} K(\bar{f}_i)K(\bar{g}_i).$$

As a consequence one can see that the difference between even and odd spin gradings in this case is $2^{g_s+b-1} = 2^{\frac{g+b-1}{2}}$.

Remark 5.21. Kasteleyn orientation are named after W. Kasteleyn, who used them to analyze dimer statistics, see for example [14]. The

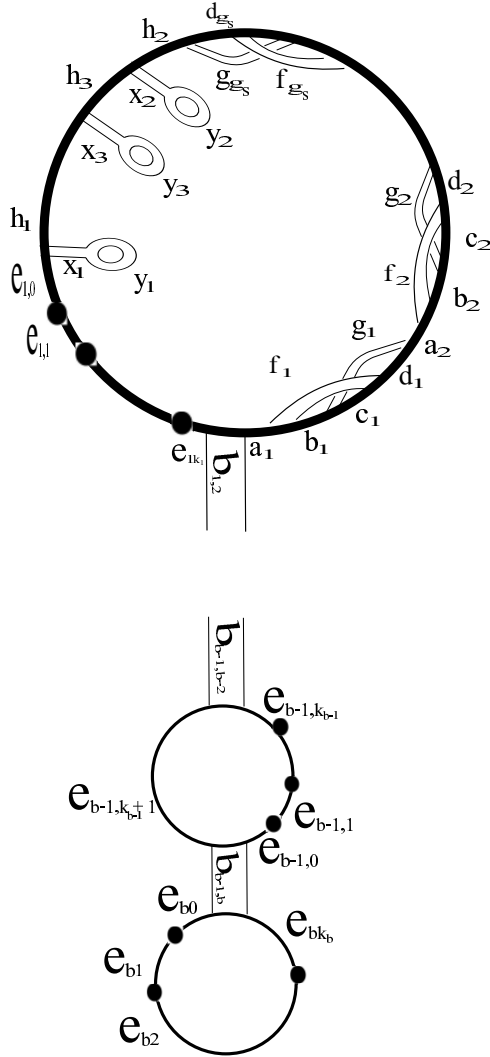


FIGURE 4.

connection between Kasteleyn orientations and spin structures on closed surfaces is obtained in [16, 6].

5.1.1. *Adjacent Kasteleyn orientations.* By Proposition 4.47, in the cell structure of $\overline{\mathcal{M}}_{g,k,l}^{comb}$, the cell $(G, [K])$ is adjacent to cells of the form $(G_e, [K_e])$, for some $e \in E(G) \setminus Br(G)$, $[K_e] \in [K(G_e)]$. We now describe $[K_e]$ explicitly in terms of $[K]$.

Fix a Kasteleyn orientation $K \in [K]$. There are two cases to consider. The first one is that e is a boundary edge which is a loop. In this case $G_e = G$, and if f is the unique edge which shares a vertex with e , define an assignment K' , by $K'(h) = K(h)$ for any h with $h/s_1 \neq f$.

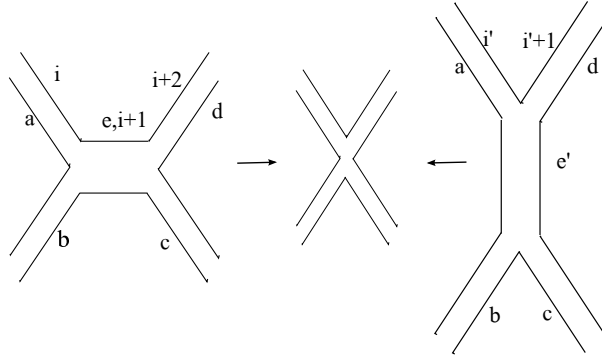


FIGURE 5. G , $\partial_e G$, and G_e .

Otherwise $K'(h) = K(h) + 1$. Consider now the complementary case. Write h for the unique half edge with $K(h) = 1, h/s_1 = e$. Write $a = s_0(h), b = s_0^2(h), c = s_1(s_0(s_1(h))), d = s_1(s_0^2(s_1(h)))$, see Figure 5. For shortness write \bar{x} for $s_1(x)$. Apart from some borderline cases which may be treated separately, we may assume all these vertices and half edges are distinct, and then, using vertex flips if needed, we may also restrict ourself to the case where $K(\bar{d}) = 1$. Write $G' = G_e$. Note that $E(G) \setminus e = E(G') \setminus e'$ canonically, for some $e' \in E(G')$. We therefore identify these sets, and also identify $H(G) \setminus \{h, s_1 h\}$ and $H(G') \setminus s_1^{-1} e'$. In G' , let v'_1 be the vertex from which a, \bar{d} issue, and v'_2 be the vertex from which b, \bar{c} issue. We may take the half edge h' to be the third half edge from v'_1 . Define the assignment $K' : H^I(G) \rightarrow \mathbb{Z}_2$ by

$K'(h') = 1, K'(\bar{h}') = 0, K'(d) = K(d) + 1 = 1, K'(\bar{d}) = K(\bar{d}) + 1 = 0$,
and $K'(f) = K(f)$ for any other half edge f .

Lemma 5.22. $K' \in [K(G')]$, and moreover $K' \in [K_e]$.

Proof. We prove only in the complementary case where e is not a boundary loop. The case where e is a boundary loop is simple and follows easily from Proposition 4.47. The first assertion is simple, we prove the second one. Write $C(G), C(G')$ for the set of closed paths without backtracking in G, G' respectively. Write $O(G), O(G')$ for the set of open directed paths without backtracking in G, G' respectively, which connect boundary vertices which are not marked points. We have bijections $f_C : C(G) \rightarrow C(G'), f_O : O(G) \rightarrow O(G')$ defined as follows. For a path $(e_1 \rightarrow e_2 \rightarrow \dots e_m) \in C(G)$, the path $f_C(e_1 \rightarrow e_2 \rightarrow \dots e_m) \in C(G')$ is defined by erasing any appearance of e in the sequence and adding e' any time we have a move $f \rightarrow f'$ where the third edge of the vertex between f and f' is e . The inverse

map is defined similarly, but with changing the roles of e, e' . The map f_O is defined in the same way.

Using Proposition 5.14 it is straight forward to verify that for any $\gamma \in C(G)$, $q_K(\gamma) = q_{K'}(f_c(\gamma))$, and for any $\gamma \in O(G)$, $Q_K(\gamma) = Q_{K'}(f_c(\gamma))$.

Now, let $(\Sigma_t, \mathbb{S}_t, s_t)_{t \in [0,1]}$ be a continuous path in $\overline{\mathcal{M}}_{g,k,l}^{comb}$, with $(\Sigma_t, \mathbb{S}_t, s_t) \in comb^{-1}(\mathcal{M}_{(G_t, z_t)})$, where

$$G_t = \begin{cases} G, & \text{if } t < \frac{1}{2}, \\ \partial_e G, & \text{if } t = \frac{1}{2}, \\ G', & \text{if } t > \frac{1}{2}, \end{cases}$$

and the graded structure $z_0 \in Z_G$ corresponds to the Kasteleyn orientation $[K]$. In light of Lemma 2.44, Proposition 5.17 and isotopy arguments, the Kasteleyn orientation on G' defined by $(\Sigma_t, \mathbb{S}_t, s_t)_{t \in (\frac{1}{2}, 1)}$ is the unique class of Kasteleyn orientation for which for any continuous family $(\gamma_t \subseteq \Sigma_t)$ of closed paths or bridges, $q(\gamma_t)$, or $Q(\gamma_t)$ is constant. By performing an isotopy, we may assume that γ_t is in fact a path in the graph G_t . It is easy to see that for ε small enough, $f_G(\gamma_{\frac{1}{2}-\varepsilon}) = \gamma_{\frac{1}{2}+\varepsilon}$, in case γ_t are closed, or $f_O(\gamma_{\frac{1}{2}-\varepsilon}) = \gamma_{\frac{1}{2}+\varepsilon}$, in case they are open. In the first case, $q_{[K]}(\gamma_{\frac{1}{2}-\varepsilon}) = q_{[K']}(\gamma_{\frac{1}{2}+\varepsilon})$, while in the second the same equation holds for Q . By Lemma 2.44, part (c), and Theorem 5.12, the graded structure $z_t, t > \frac{1}{2}$ must correspond to $[K']$. \square

5.1.2. Trivalent graphs.

Definition 5.23. Let G be a trivalent graph. Recall that a half node is a $(N^B)^{-1}$ -preimage of a node, and that their collection is denoted $HN(G)$. An *extended Kasteleyn orientation* on G is a map $K : H(G) \cup HN(G) \rightarrow \mathbb{Z}_2$, which satisfies

- (a) For any $h \in H^B$, $K(h) = 0$.
- (b) For any $h \in H$, $K(h) + K(s_1 h) = 1$.
- (c) For any node v , if $|N^{-1}(v)| = 3$, then $K|_{N^{-1}(v)} = 1$. Otherwise $K(v_{i,1}) + K(v_{i,2}) = 1$, where $N^{-1}(v) = \{v_{i,1}, v_{i,2}\}$.
- (d) For any face f , $\sum K(x) = 1$, where the variable x is taken from the set of half edges with $x/s_2 = f$, together with the set of half nodes which belong to f .

Two extended Kasteleyn orientations are equivalent if they differ by the action of internal vertex flips. Write $[K]$ for the equivalence class of K . Define $K(G), [K(G)]$ as the sets of extended Kasteleyn orientations and the set of equivalence classes of extended Kasteleyn orientations.

Write $\text{Aut}(G, [K])$ as the automorphism subgroup of G which preserves $[K]$.

With the same exact techniques of Subsection 5.1, together with Corollary 2.22, we obtain

Lemma 5.24. *For a trivalent G and a metric ℓ , there is a natural bijection between $\text{Spin}((\text{comb}^{\mathbb{R}})^{-1}(G, \ell))$ and $[K(G)]$. The induced map $\coprod_{z \in Z_G/\text{Aut}(G)} \mathcal{M}_{(G,z)} \rightarrow \coprod_{[K] \in [K(G)]/\text{Aut}(G)} \mathbb{R}_+^{E(G)}/\text{Aut}(G, [K])$, is a homeomorphism. In particular, $Z_G \simeq [K(G)]$ canonically. A half node v in (G, z) is illegal if and only if $K(v) = 1$ for any $K \in [K]$ which corresponds to z .*

From now on we denote trivalent graphs (G, z) by $(G, [K])$, for the corresponding $[K] \in [K(G)]$.

Definition 5.25. Define $\mathcal{M}_{(G,[K])} = \mathbb{R}_+^{E(G)}/\text{Aut}(G, [K])$, the moduli of metrics on \mathcal{M}_G , together with a fixed equivalence class of Kasteleyn orientations. Define $\overline{\mathcal{M}}_{(G,[K])} = \mathbb{R}_{\geq 0}^{E(G)}/\text{Aut}(G, [K])$. For $f_1, \dots, f_s \in E(G)$, set $\partial_{f_1, \dots, f_s} \overline{\mathcal{M}}_{(G,[K])}$ as the face of $\overline{\mathcal{M}}_{(G,[K])}$ defined by setting the coordinates f_1, \dots, f_s to 0. For p_1, \dots, p_l define $\mathcal{M}_{(G,[K])}(\mathbf{p}), \overline{\mathcal{M}}_{(G,[K])}(\mathbf{p})$.

Suppose G is a trivalent graph $K \in K(G)$, and $e \in \text{Br}(G)$. In case e is a boundary edge, let h_1 be its internal half edge, $h/s_1 = e, h \in H^I$. In case e is an internal edge, write $s_1^{-1}(e) = \{h_1, h_2\}$, where $K(h_i) = i(\text{mod } 2)$. Define $\partial_e K$ to be the unique map $\partial_e K : H(\partial_e G) \cup HN(\partial_e G) \rightarrow \mathbb{Z}_2$, which equals K on any half edge $h' \notin s_1^{-1}e$, and $\partial_e K(\partial_e h_i) = i(\text{mod } 2)$. In a similar way, one can define $\partial_{e_1, \dots, e_r} K$ for a compatible sequence of bridges.

It is straight forward that

Observation 5.26. For any trivalent $(G, [K])$, and a bridge e , the graph $(\partial_e G, [\partial_e K])$ is a well defined trivalent graph, in particular $\partial_e K \in [K(\partial_e G)]$. Moreover, $\partial_e : [K(G)] \rightarrow [K(\partial_e G)]$ is a bijection.

In addition, for any trivalent connected graph $(G, [K])$, there is a unique smooth trivalent $(G', [K'])$, and a unique, up to order, compatible sequence of bridges e_1, \dots, e_r with $(G, [K]) = \partial_{e_1, \dots, e_r}(G', [K'])$.

With the same techniques of the proof of Lemma 5.22 one obtains

Lemma 5.27. *Let G be a trivalent graph, e_1, \dots, e_r a compatible sequence of bridges. Under the identification of Lemma 5.24 between $Z_H, [K(H)]$, for $H = G, \partial_{e_r} G, \dots, \partial_{e_1, \dots, e_r} G$,*

$$\overline{\mathcal{M}}_{\partial_{e_1, \dots, e_r}(G, [K])} \hookrightarrow \partial_{e_1, \dots, e_r} \overline{\mathcal{M}}_{\partial_{e_{s+1}, \dots, e_r}(G, [K])},$$

canonically.

We shall therefore identify $\overline{\mathcal{M}}_{(G,z)}$ and the corresponding $\overline{\mathcal{M}}_{(G,[K])}$ without further notice.

5.2. Orientation. In this subsection we construct an orientation to $\overline{\mathcal{M}}_{g,k,l}^{comb}$. We do it by writing an explicit formula for the orientation of each highest dimensional cell of $\overline{\mathcal{M}}_{g,k,l}^{comb}(\mathbf{p})$, that is, for cells $\mathcal{M}_{(G,[K])}(\mathbf{p})$ where $G \in \mathcal{R}^0$, $[K] \in [K(G)]$, and then showing that on codimension 1 faces between two such cells, the induced orientations disagree. We also discuss the induced orientation on the boundary, and prove that these orientations are the ones induced from $\overline{\mathcal{M}}_{g,k,l}$ by $comb_*$.

For $G \in \mathcal{R}_{g,k,l}^0$, we have a map

$$(24) \quad A_G : \mathbb{R}_+^{E(G)} \rightarrow \mathbb{R}^{F(G)} = \mathbb{R}^{[l]},$$

which takes a collection of edge length and returns the face perimeters. $\mathcal{M}_{(G,[K])}(\mathbf{p}) = A_G^{-1}(\mathbf{p})/Aut(G, [K])$. In particular, orienting $\mathcal{M}_{(G,[K])}(\mathbf{p})$ is equivalent to orienting $ker(A_G)/Aut(G, [K])$. Using the exact sequence

$$(25) \quad 0 \rightarrow ker(A_G) \rightarrow \mathbb{R}^{E(G)} \rightarrow \mathbb{R}^{F(G)} = \mathbb{R}^{[l]} \rightarrow 0,$$

we see that orienting $\mathbb{R}^{E(G)}, \mathbb{R}^{[l]}$, or equivalently, ordering $E(G), [l]$, up to even permutations, gives an orientation to $\mathcal{M}_{(G,[K])}(\mathbf{p})$, as long as the action of $Aut(G, [K])$ preserves the orientation.

Fix any order for $[l]$, for example $1, 2, \dots, l$. Choose any Kasteleyn orientation $K \in [K]$. Define $\mathfrak{o}_i = \mathfrak{o}_{(G,K,i)}$ by

$$\bigwedge_{K(h)=1, h/s_2=i} d\ell_h,$$

that is, we take the wedge of $d\ell_h$ over half edges h of face i , with $K(h) = 1$. The wedge is taken counterclockwise. Because there is an odd number of half edges of the i^{th} face with $K = 1$, the element \mathfrak{o}_i is well defined, and independent on which half edge appears first. In addition, \mathfrak{o}_i is an odd variable.

Definition 5.28. Choose any Kasteleyn orientation K . Put

$$\mathfrak{o}_{(G,K)} = \bigwedge_{i=1}^l \mathfrak{o}_i.$$

Define $\bar{\mathfrak{o}}_{(G,K)}$ as the orientation on $ker(A_G)$ induced from exact sequence 25, when $\mathbb{R}^{E(G)}$ is oriented by $\mathfrak{o}_{(G,K)}$, and $\mathbb{R}^{[l]}$ by $\bigwedge_{i=1}^l dp_i$.

Remark 5.29. Because both dp_i and \mathfrak{o}_i are odd variables, choosing another order on $[l]$ does not change $\bar{\mathfrak{o}}_G$.

Lemma 5.30. $\bar{\mathfrak{o}}_{(G,K)}$ depends only on $[K]$.

Notation 5.31. Let n be a good ordering, as in Definition 5.1, and $K \in K(G)$ a Kasteleyn orientation. Define $H_K = \{h \in H^I \mid K(h) = 1\}$. We also define $n_K : |H^I| \rightarrow \mathbb{Z}$ by

$$n_K(i) = |\{h \in H_K \mid n(h) < i\}|.$$

Note that the restriction of a good ordering to a subset of H^I induces an order on its elements.

Proof of Lemma 5.30. Take any $K \in K(G)$. We recall from Lemma 5.9 that any other element of $K(G)$ can be obtained from K by successive flips in vertices. It will thus suffice to prove that the orientations induced by K, K' are the same when K, K' differ by a single flip in vertex v . It will be enough to prove that $\mathfrak{o}_{(G,K)} = \mathfrak{o}_{(G,K')}$

Fix a good ordering n . By definition

$$\mathfrak{o}_{(G,K)} = \bigwedge_{e \in H_K} d\ell_e,$$

where the order of the wedging is the order n restricted to H_K . The sign difference between $\mathfrak{o}_{(G,K)}, \mathfrak{o}_{(G,K')}$ can be found geometrically by the following procedure. Define

$$L_K = \{(n(h), 0) \mid h \in H_K\}, L_{K'} = \{(n(h), 1) \mid h \in H_{K'}\} \subseteq \mathbb{R}^2.$$

For any $e \in E$ draw the chord $c(e)$ between $(n(h_0), 0) \in L_K, (n(h_1), 1) \in L_{K'}$ where $h_0/s_1 = h_1/s_1$. By definition the change of signs between $\mathfrak{o}_{G,K}, \mathfrak{o}_{G,K'}$ is just the parity of the number of intersections of these chords (slightly perturbed, if necessary). We shall prove that this number is always even. Note that for all edges except for those issuing from v , the chords are parallel and vertical.

Let h_1 be an half edge of v . Put $h_2 = s_0(h_1), h_3 = s_0^2(h_1)$, and $\bar{h}_j = s_1(h_j)$. Apart from some borderline cases which can be treated separately, we may assume that we are in the following scenario,

$$\begin{aligned} n(\bar{h}_2) &= i_1, n(h_1) = i_1 + 1, n(\bar{h}_3) = i_2, \\ n(h_2) &= i_2 + 1, n(\bar{h}_1) = i_3, n(h_3) = i_3 + 1. \end{aligned}$$

Thus, the chord c_{h_j} is either the chord between $(i_j + 1, 0)$ and $(i_{j-1}, 1)$, or the chord between $(i_j + 1, 1)$ and $(i_{j-1}, 0)$. It is easy to see that the number of vertical chords it intersects is the size of

$$I_j = \{h \in H_K \setminus \{h_i, \bar{h}_i\}_{i=1,2,3} \mid n(h) \in (a_j, b_j)\},$$

where $a_j = \min(n_K(i_j + 1), n_K(i_{j-1}))$, $b_j = \max(n_K(i_j + 1), n_K(i_{j-1}))$. For exactly one $j \in \{1, 2, 3\}$ we have $I_j = I_{j+1} \cup I_{j+2}$, where addition is modulo 3, and the union is disjoint. Thus, any vertical chord either

misses the chords c_{h_j} or meets exactly two of them. In addition, it can be checked directly that the chords c_{h_j} intersect each other an even number of times. And the lemma follows. \square

Corollary 5.32. *For any $G \in \mathcal{R}_{g,k,l}^0$, $[K] \in [K(G)]$, the group $\text{Aut}(G, [K])$ acts in an orientation preserving manner. In particular, the orientation $\bar{\mathbf{o}}_{(G,K)}$ induces, for any \mathbf{p} an orientation on $\mathcal{M}_{(G,[K])}$.*

Denote this orientation by $\bar{\mathbf{o}}_{(G,[K])}$. The main theorem of this subsection is

Theorem 5.33. *The orientations $\bar{\mathbf{o}}_{(G,[K])}$ induce a canonical orientation on the space $\overline{\mathcal{M}}_{g,k,l}^{\text{comb}}(\mathbf{p})$.*

Proof. We shall show that the orientations \mathbf{o}_G for $G \in \mathcal{SR}_{g,k,l}^0$ are compatible on codimension 1 faces. This will show that a suborbifold of $\mathcal{M}_{g,k,l}^{\text{comb}}$, which differs from $\mathcal{M}_{g,k,l}^{\text{comb}}$ in codimension 2 cells is oriented, hence also $\mathcal{M}_{g,k,l}^{\text{comb}}$. Since $\mathcal{M}_{g,k,l}^{\text{comb}}$ differs from $\overline{\mathcal{M}}_{g,k,l}^{\text{comb}}$ by codimension 2 strata in the interior, and in codimension 1 boundary, this argument will show that $\overline{\mathcal{M}}_{g,k,l}^{\text{comb}}$ is also endowed with a canonical orientation.

We therefore have to show that for any $(G, [K]) \in \mathcal{SR}_{g,k,l}^0$, $e \in E(G) \setminus \text{Br}(G)$, $(G', [K']) = (G_e, [K]_e)$ the induced orientation on $\partial_e \mathcal{M}_{(G,[K])}$ once by $\mathcal{M}_{(G,[K])}$ and once by $\mathcal{M}_{(G,[K])}$ disagree. The case where e is a boundary loop is a special case of the proof of Lemma 6.21. We move to the general case.

Put $H^I = H^I(G)$, $H'^I = H^I(G')$. Note that we have a natural identification of $E(G) \setminus e$ and $E(G') \setminus e'$, for some edge e' , so from now on we treat them as the same set. Choose an good ordering n for H^I . There exists a good ordering n' of H'^I , which, when restricted to $H'^I \setminus \{s_1^{-1}e'\}$, defines the same order as the restriction of n to $H^I \setminus \{s_1^{-1}e\}$. Fix a Kasteleyn orientation $K \in K(G)$, set $h \in s_1^{-1}e$ with $K(h) = 1$. Write $a = s_0(h)$, $b = s_0^2(h)$, $c = s_1(s_0(s_1(h)))$, $d = s_1(s_0^2(s_1(h)))$, see Figure 5. For shortness write \bar{x} for $s_1(x)$. Apart from some borderline cases which may be treated separately, we may assume all these vertices and half edges are distinct, and then, using vertex flips if needed, we may also restrict ourself to the case where $K(\bar{d}) = 1$. In this case we can assume n was chosen in such a way that

$$\begin{aligned} n(\bar{a}) &= i, \quad n(h) = i + 1, \quad n(\bar{d}) = i + 2, \\ n(d) &= m, \quad n(\bar{c}) = m + 1, \\ n(c) &= h, \quad n(\bar{h}) = h + 1, \quad n(b) = h + 2, \\ n(\bar{b}) &= j, \quad n(a) = j + 1. \end{aligned}$$

as in Figure 5.

A canonical outward normal for $\mathcal{M}_{\partial_e G} \hookrightarrow \overline{\mathcal{M}}_G$ is just $-d\ell_e$. We see that the induced orientation on $\mathcal{M}_{\partial_e G}$ is just

$$(26) \quad (-1)^{n_K(n(h))+1} \bigwedge_{f \in H_K \setminus \{h\}} d\ell_f = (-1)^{n_K(i+1)+1} \bigwedge_{f \in H_K \setminus \{h\}} d\ell_f,$$

where as usual the wedge is taken in the order on n_K induced by n .

In G' , let v'_1 be the vertex from which a, \bar{d} issue, and v'_2 be the vertex from which b, \bar{c} issue. We may take the half edge h' to be the third half edge from v'_1 . Then, for some i', m', h', j' we have

$$\begin{aligned} n'(\bar{a}) &= i', \quad n'(\bar{d}) = i + 1, \\ n'(d) &= m, \quad n'(h') = m' + 1, \quad n'(\bar{c}) = m' + 2, \\ n'(c) &= h, \quad n'(b) = h' + 1, \\ n'(\bar{b}) &= j', \quad n'(\bar{h}') = j' + 1, \quad n'(a) = j' + 2. \end{aligned}$$

By Lemma 5.22 we have a representative K' of $[K]_e$, described by

$$K'(h') = 1, \quad K'(\bar{h}') = 0, \quad K'(d) = K(d) + 1 = 1, \quad K'(\bar{d}) = K(\bar{d}) + 1 = 0,$$

and $K'(f) = K(f)$ for any other half edge f . As above, a canonical outward normal for $\mathcal{M}_{\partial_{e'} G'} \hookrightarrow \overline{\mathcal{M}}_{G'}$ is just $-d\ell_{e'}$. We see that the induced orientation on $\mathcal{M}_{\partial_{e'} G'}$ is just

$$(27) \quad (-1)^{n_{K'}(n'(h'))+1} \bigwedge_{f \in H_{K'} \setminus \{h'\}} d\ell_f = (-1)^{n_{K'}(m'+1)+1} \bigwedge_{f \in H_{K'} \setminus \{h'\}} d\ell_f.$$

The choice of n, n', K' , makes the terms $\bigwedge_{f \in H_K \setminus \{h\}} d\ell_f, \bigwedge_{f \in H_{K'} \setminus \{h'\}} d\ell_f$ differ only in the relative location of $d\ell_d$. By our assumptions on $K(\bar{d}), K'(\bar{d})$ the difference is just the difference between $n_K(\bar{d}) - 1 = n_K(i + 2) - 1$ and $n'_{K'}(d) = n'_{K'}(m')$. We subtracted 1 from $n_K(\bar{d})$ because we did not want to count h which occurs before \bar{d} in the order n . Now, $n_K(i + 2) - 1 = n_K(i + 1)$, as $n(h) = i, K(h) = 1$. Similarly, $n'_{K'}(m') = n'_{K'}(m' + 1) - 1$, since $n'(d) = m', K'(d) = 1$.

The total difference between the two orientations is thus

$$(-1)^{n'_{K'}(m'+1)+1+n'_{K'}(m'+1)-1+n_K(i+1)+1+n_K(i+1)} = -1,$$

as claimed. \square

Remark 5.34. The spaces $\mathcal{M}_{g,k,l}, \mathcal{M}_{g,k,l}^{comb}(\mathbf{p})$ are homeomorphic, therefore the last theorem gives, in fact, another proof that $\overline{\mathcal{M}}_{g,k,l}$ is oriented. Later we shall see that the orientation constructed here agrees with the orientation of [21].

Corollary 5.35. *For $G \in \mathcal{SR}_{g,k,l}^0$ and an internal edge e which is not a bridge, the orientations on $\partial_e \overline{\mathcal{M}}_{(G,[K])}(\mathbf{p}) \simeq \partial_e \overline{\mathcal{M}}_{(G_e,[K_e])}(\mathbf{p})$, induced as boundaries of $\mathcal{M}_{(G,[K])}(\mathbf{p})$, $\mathcal{M}_{(G_e,[K_e])}(\mathbf{p})$ are opposite.*

5.3. Critical nodal graphs and their moduli.

5.3.1. *Critical nodal ribbon graphs.* In this subsection we describe critical nodal graphs. They will parameterize strata which will contribute in the combinatorial formula. For completeness we first describe slightly more general graphs.

Definition 5.36. A *graded nodal ribbon graph* is a graded ribbon graph (G, z) , together with a subset $\mathcal{V}(G)$ of legal points in $B(\text{Norm}(G)) \setminus B(G)$. We call $\mathcal{V}(G)$ the set of *legal nodes* of the nodal graph and $s_1 \mathcal{V}(G)$ the illegal nodes, where s_1 was defined in Notation 4.26. The vertices and edges of the nodal graph are the vertices and edges of $\text{Norm}(G, z)$ after forgetting the illegal markings $s_1 \mathcal{V}(G)$. A metric is a metric on these edges. If e is an edge in the nodal graph, $\partial_e(G, z, \mathcal{V})$ is the nodal graph with underlying graph $\partial_e(G, z)$ and legal nodes are those legal nodes in $\partial_e(G, z)$ which remain special points in $\text{Norm}(\partial_e(G, z))$ after the contraction, where we use the natural correspondence between special points in $\text{Norm}(G, z)$ and in $\text{Norm}(\partial_e(G, z))$.

The *components* of the nodal graph are the connected components created after removing $s_1 \mathcal{V}(G)$. More precisely, define an equivalence relation \sim_N on the components of $\text{Norm}(G, z)$ as follows. Components $C_1, C_2 \in H_0(\text{Norm}(G, z))$ are neighbours if one of them contains a legal point $u \notin \mathcal{V}(G)$ such that $s_1 u$ belong to the other component. We write $C_1 \sim_N C_2$, for $C_1, C_2 \in H_0(\text{Norm}(G, z))$, if they can be connected in a path of neighboring components. The components of the nodal graph are defined to be the *Norm*–image of \sim_N –equivalence classes.

In case the underlying graph is effective we have a more convenient definition.

Definition 5.37. An *effective graded nodal ribbon graph* is a tuple $(G_i, z_i, m, \{\mathcal{V}_e\})$, or (G, z) for shortness, where

- (a) (G_i, z_i) is an effective graded ribbon graph.
- (b) $m : \bigcup_i s_1 H^B(G_i) \rightarrow \mathbb{Z}_{\geq 0}$.
- (c) $\mathcal{V}_e : [m(e)] \rightarrow \bigcup_i B(G_i)$, $e \in \bigcup_i s_1 H^B(G_i)$ are injections.

We require the sets $\mathcal{V}_e = \mathcal{V}_e([m(e)])$ to be disjoint.

Let G be the graph obtained by choosing $m(e)$ points $p_{e,1}, \dots, p_{e,m(e)}$ on e , ordered according to the orientation of the boundary and identifying $p_{e,i}$ with $\mathcal{V}_e(i)$. $C(G_i, z_i, m, \{\mathcal{V}_e\})$ denotes the different graded components of the graph, that is the collection of (G_i, z_i) .

Write $E(G) = \cup_i E(G_i)$, similarly define $H^I(G), H^B(G), V(G), F(G)$. For a boundary edge $e = h/s_1$, where $h_1/ \in s_1 H^B$ we sometimes write $m(e) = m(h)$. Vertices in the image of \mathcal{V}_e are called legal nodes and their set is denoted by $\mathcal{V}(G)$. The boundary marked points of G are boundary marked points of G_i 's which are not legal nodes. Denote them by $B(G)$. Define $I(G) = \cup I(G_i)$.

An effective graded nodal ribbon graph is naturally embedded into the (topological) nodal surface $\Sigma = (\coprod_i \Sigma_i) / \sim$, defined as follows. Σ_i is the topological open marked surface to which G_i embeds, and in case G_i is a ghost it is a point. We identify G_i as a subspace of Σ_i . We add $m(e)$ points on the edge e , $p_{e,1}, \dots, p_{e,m(e)}$, and quotient by $p_{e,i} \sim \mathcal{V}_e(i)$. The genus of the graph is defined to be the (doubled) genus of Σ .

A *marked effective nodal graph* is an effective nodal graph together with markings $m^B : B(G) \rightarrow \mathbb{Z}$, $m^I : I(G) \rightarrow \mathbb{Z}$.

A *graded critical nodal ribbon graph* is a nodal graph such that each $(G_i, z_i) \in \mathcal{SR}^0$. In this case we use the Kasteleyn notation for components, $(G_i, [K_i])$ rather than (G_i, z_i) , and we denote the whole graph by $(G, [K])$ for shortness.

A graded critical nodal graph G is *odd*, if each $G_i \in \tilde{\mathcal{SR}}^0$.

The notion of an isomorphism is the expected one. Write $\mathcal{SR}_{g,k,l}^m$ for the collection of isomorphism classes of marked critical nodal graded ribbon graphs G with m nodes, genus g , such that $m^B : B(G) \simeq [k]$, $m^I : I(G) \simeq [l]$. Let $\tilde{\mathcal{SR}}_{g,k,l}^m$ be the subset of such graphs which are odd. Write $\text{Aut}(G, [K])$ for the group of automorphisms of $(G, [K]) \in \mathcal{SR}_{g,k,l}^m$.

Define non graded critical nodal ribbon graphs $G = (G_i, m, \{\mathcal{V}_e\})$, in the same way, only without the data of Kasteleyn orientations, so that $G_i \in \mathcal{R}^0$, rather than \mathcal{SR}^0 . Denote by $\mathcal{R}_{g,k,l}^m$ the collection of isomorphism classes non graded critical nodal ribbon graphs G with m nodes, genus g , such that $m^B : B(G) \simeq [k]$, $m^I : I(G) \simeq [l]$. Let $\tilde{\mathcal{R}}_{g,k,l}^m$ be the subset of such graphs which are odd. Write $\text{Aut}(G)$ for the group of automorphisms of $G \in \mathcal{R}_{g,k,l}^m$.

A metric on a graded critical nodal ribbon graph is an assignment of positive length to its edges.

A bridge $e \in E(G)$ is an edge which is a bridge in one component G_i of G . An effective bridge is a bridge with $m(e) = 0$, when m is defined. Let $\text{Br}(G, [K])$ to be the collection of bridges, and $\text{Br}^{eff}(G, [K])$ the collection of effective bridges. As in the non nodal case, for shortness we shall usually omit $[K]$ from the notations of $\text{Br}, \text{Br}^{eff}$.

When it is understood from context whether or not the nodal graph is graded, we omit the words graded\non graded, and just say critical nodal.

Remark 5.38. It is simple to verify that when (G, z) is effective the two definitions 5.36, 5.37 agree. We shall therefore use Definition 5.37 whenever possible. It is also straightforward to verify that the definition of $\tilde{\mathcal{R}}_{g,k,l}^m$ agrees with the one given in 1.4. Note that in a metric critical nodal ribbon graph we do not know distance from illegal nodes to other vertices. We do know on which edge an illegal node lays, and relative order of illegal nodes on this edge. See the example in Figure 1.

Observation 5.39. Under $for_{spin} : \mathcal{SR}_{g,k,l}^m \rightarrow \mathcal{R}_{g,k,l}^m$, which forgets the Kasteleyn orientation, odd graphs go to odd graphs, and the preimage of G is canonically $[K(G)]/Aut(G)$.

5.3.2. Trivalent graphs versus graded critical nodal graphs. Given a connected trivalent ribbon graph $(G, [K])$, we define a graded critical nodal graph $\mathcal{Y}(G, [K])$. Its components are the components of $Norm(G, [K])$, after erasing every illegal boundary point and concatenating its two edges to one edge. Suppose e is an edge obtained by concatenating e_1, \dots, e_{m+1} in the described process, and in this order. Define $m(e) = m$. Suppose v_i is the vertex between e_i, e_{i+1} , then $\mathcal{V}_e(i) = s_1 v_i$, where we use Notation 4.26. It is easy to verify that

Observation 5.40. The map \mathcal{Y} is a bijection between connected trivalent graphs and graded critical nodal ribbon graphs. For any connected trivalent $(G, [K])$ there is a bijection between bridges in $(G, [K])$ and effective bridges in $\mathcal{Y}(G, [K])$.

We now extend the definition of \mathcal{Y} to metric graphs. For a connected trivalent metric graph $(G, [K], \ell)$ define the graded critical nodal metric graph $\mathcal{Y}(G, [K], \ell) = (\mathcal{Y}(G, [K]), \mathcal{Y}\ell)$, by $\mathcal{Y}\ell_e = \ell_e$, if the edge e is an edge of $Norm(G, [K])$. Otherwise, if e is the union of e_1, \dots, e_{m+1} , define $\mathcal{Y}\ell_e = \sum_{i=1}^{m+1} \ell_{e_i}$. Note that the perimeters are left unchanged.

Notation 5.41. Suppose $(G, [K]) \in \mathcal{SR}_{g,k,l}^m(\mathbf{p})$, $e = \{h_1, h_2 = s_1 h_1\} \in Br^{eff}(G)$, with $K(h_1) = 0$. Define the graph $\mathcal{B}_e(G, [K]) = \mathcal{B}\partial_e(G, [K])$ as follows. Suppose $G = \cup_{i \in [n]} G_i$. Write $v_i = \partial_e(h_i)$, the vertex obtained by contracting h_i in $\partial_e G$. Without loss of generality assume e is an edge of component G_n . Write $x = s_2 h_1, y = s_1(s_2^{-1} h_1) \in H^I(\partial_e G_n)$.

Define the graph $\mathcal{B}_e G$ to be the graph whose first $n-1$ components, G'_i are just $G_i, i \leq n-1$. $K'_i = K_i, m' = m, \mathcal{V}'_e = \mathcal{V}_e$ for these components.

In case the normalization $Norm(\partial_e G_n)$ is disconnected, let G'_n be the component which does not contain v_2 , K', m', \mathcal{V}' will be the induced maps. Note that G'_n may be a ghost. Define the component G'_{n+1} as the graph obtained by the component of v_2 in $Norm(\partial_e G_n)$, after gluing the half edges $x/s_1, y/s_1$ to a new edge xy , and removing the vertex v_2 . The updated Kasteleyn orientation is the unique Kasteleyn orientation which gives any internal half edge its value under K_n . For any half edge $e' \neq xy$, $m'(e') = m(e')$, $m(xy) = m(x) + m(y) + 1$. Similarly $\mathcal{V}'(e') = \mathcal{V}(e')$ for $e' \neq xy$, while

$$(28) \quad \mathcal{V}'_{xy}(a) = \begin{cases} \mathcal{V}_y(a), & a \leq m(y) \\ v_1, & a = m(y) + 1 \\ \mathcal{V}_x(a - m(y) - 1), & a > m(y) + 1. \end{cases}$$

If $\partial_e G_n \setminus \{v_e\}$ is connected, set G'_n to be the component of v_1 in the normalization, where again edges x, y are glued and v_2 is removed, and K', m', \mathcal{V}' are defined in the same way as above.

There is a canonical surjection, which we shall also denote by \mathcal{B}_e ,

$$E(G) \cup \mathcal{V}(G) \rightarrow E(\mathcal{B}_e G) \cup \mathcal{V}(\mathcal{B}_e G).$$

It takes e to v_1 , and all other edges to the corresponding edges, so that it is one to one, except on the edges x, y which go to xy .

With the same notations, given a metric ℓ on the graph, with $\ell_e = 0$, the graph $\mathcal{B}_e(G, [K], \ell)$ is the graded critical nodal graph with underlying graph $\mathcal{B}_e(G, [K])$, and the metric $(\mathcal{B}_e \ell)_{e'} = \ell_{e'}$ for $e' \neq x, y$, and $\mathcal{B}_e \ell_{xy} = \ell_x + \ell_y$. For convenience we usually denote $\mathcal{B}\ell$ by ℓ as well.

A *compatible sequence of effective bridges*, e_1, \dots, e_r is a sequence of bridges such that e_{i+1} is an effective bridge in $\mathcal{B}_{e_i} \dots \mathcal{B}_{e_1} G$ for all i . For such a sequence define $\mathcal{B}_{e_1, \dots, e_r}(G, [K], \ell) = \mathcal{B}_{e_r} \dots \mathcal{B}_{e_1}(G, [K], \ell)$, and the map $\mathcal{B}_{e_1, \dots, e_r} = \mathcal{B}_{e_r} \circ \dots \circ \mathcal{B}_{e_1}$.

The next observation follows easily from Observations 5.40 and 5.26.

Observation 5.42. If $(G, [K]) \in \mathcal{SR}_{g,k,l}^m$ and $e \in Br^{eff}(G)$, then $\mathcal{B}_e G \in \mathcal{SR}_{g,k,l}^{m+1}$. Moreover, for any $(G, [K]) \in \mathcal{SR}_{g,k,l}^{m+1}$, and any legal node v , there exists a unique graph $(H, [K']) \in \mathcal{SR}_{g,k,l}^m$ and an edge $e \in Br^{eff}(H)$ with $\mathcal{B}_e(H, [K']) = (G, [K])$, and $\mathcal{B}_e e = v$.

In addition, if $(G, [K])$ is connected trivalent, $e \in Br(G, [K])$

$$\mathcal{V}(\partial_e(G, [K])) = \mathcal{B}_e(\mathcal{V}(G, [K])),$$

where we use the identification of bridges of Observation 5.40.

Notation 5.43. Recall notation 4.5. For $(G, [K]) \in \mathcal{SR}_{g,k,l}^{m+1}$, denote by $\mathcal{B}_{h,a}^{-1}(G, [K]) = \mathcal{B}_{[h],a}^{-1}(G, [K])$ the isomorphism class of triples

$(H, [K'], e)$, where $H \in \mathcal{SR}_{g,k,l}^m$, $\mathcal{B}_e(H, [K']) = (G, [K])$, $\mathcal{B}_e e = \mathcal{V}_h(a)$, for $h \in s_1(H^B(G))$, and $a \in [m(h)]$. Let

$$\mathcal{B}^{-1}G = \{\mathcal{B}_{[h],a}^{-1}(G, [K]) \mid [h] \in [s_1(H^B(G))], a \in [m(h)]\}.$$

5.3.3. Moduli of critical nodal graphs.

Definition 5.44. For any nodal ribbon graph G define $\overline{\mathcal{M}}_G$ to be the moduli of nonnegative metrics on G . In particular, given $(G, [K]) \in \mathcal{SR}_{g,k,l}^m$, define $\overline{\mathcal{M}}_{(G,[K])} = \mathbb{R}_{\geq 0}^{E(G)} / \text{Aut}(G, [K])$, and $\overline{\mathcal{M}}_{(G,[K])}(\mathbf{p})$ as the subsimplex for which the i^{th} perimeter equals $p_i > 0$. In addition, define $\mathcal{M}_{(G,[K])} = \mathbb{R}_+^{E(G)} / \text{Aut}(G, [K]) \hookrightarrow \overline{\mathcal{M}}_{(G,[K])}$, and $\mathcal{M}_{(G,[K])}(\mathbf{p})$ as the subsimplex for which all edges lengths are positive. For $e \in E(G)$, write $\partial_e \overline{\mathcal{M}}_{(G,[K])}$ to be the face of $\overline{\mathcal{M}}_{(G,[K])}$ where the edge e is of length 0. More precisely,

$$\partial_e \overline{\mathcal{M}}_{(G,[K])} = \left(\bigcup_{f \in [e]} \{\ell : E \rightarrow \mathbb{R}_{\geq 0} \mid \ell_f = 0\} \right) / \text{Aut}(G, [K]).$$

$\partial \overline{\mathcal{M}}_{(G,[K])}$ is the boundary of the simplex, which can also be written as $\bigcup_{[e] \in [E(G)]} \partial_e \overline{\mathcal{M}}_{(G,[K])}$. We similarly define $\partial_{e_1, \dots, e_r} \overline{\mathcal{M}}_{(G,[K])}$.

The map $\mathcal{B}_{e_1, \dots, e_r}$ on metric graphs induces $\mathcal{B}_{e_1, \dots, e_r} : \partial_{e_1, \dots, e_r} \overline{\mathcal{M}}_{(G,[K])} \rightarrow \overline{\mathcal{M}}_{\mathcal{B}_{e_1, \dots, e_r}(G,[K])}$. When e_1, \dots, e_r are understood from the context we denote the map by \mathcal{B} only.

Note that $\overline{\mathcal{M}}_{(\partial_e G, [\partial_e K])} = \partial_e \overline{\mathcal{M}}_{(G,[K])}$. Whenever $(G, [K]) \in \mathcal{SR}_{g,k,l}^m$ the graph $\partial_e(G, [K])$ has no automorphism, as the contracted e must be fixed.

Definition 5.45. For $(G, [K]) \in \mathcal{SR}_{g,k,l}^m$, $i \in [l]$ define the S^1 -orbibundle $\mathcal{F}_i \rightarrow \overline{\mathcal{M}}_{(G,[K])}$ to be the set of pairs (ℓ, x) where $\ell \in \overline{\mathcal{M}}_{(G,[K])}$, x is a point on the i^{th} face, with the natural topology. For a (d, l) -set L , write $S_L \rightarrow \overline{\mathcal{M}}_{(G,[K])}$ to be the sphere bundle associated to $\{S_{L(i)} \mid i \in [d]\}$, as in Notation 1. We define the forms α_i, ω_i and the other forms of Definition 4.8 as the pull-backs of the corresponding forms defined on the component which contains face i .

Observation 5.46. For any connected trivalent graph $(G, [K])$, the maps

$$\mathcal{Y} = \mathcal{Y}^{(G,[K])} : \overline{\mathcal{M}}_{(G,[K])} \rightarrow \overline{\mathcal{M}}_{\mathcal{Y}(G,[K])}, \overline{\mathcal{M}}_{(G,[K])}(\mathbf{p}) \rightarrow \overline{\mathcal{M}}_{\mathcal{Y}(G,[K])}(\mathbf{p})$$

defined by the map \mathcal{Y} on metric graphs, are piecewise linear submersions. Moreover, for any $e \in \text{Br}(G, [K])$, the following diagram commutes,

$$\begin{array}{ccccc} \mathcal{M}_{\partial_e(G, [K])} & \xlongequal{\quad} & \partial_e \overline{\mathcal{M}}_{(G, [K])} & \xhookrightarrow{\quad} & \overline{\mathcal{M}}_{(G, [K])} \\ \downarrow \mathcal{Y}^{\partial_e(G, [K])} & & \downarrow \mathcal{Y}^{(G, [K])} & & \downarrow \mathcal{Y}^{(G, [K])} \\ \mathcal{M}_{\mathcal{B}_e \mathcal{Y}(G, [K])} & \xleftarrow{\quad \mathcal{B}_e \quad} & \partial_e \overline{\mathcal{M}}_{\mathcal{Y}(G, [K])} & \xhookrightarrow{\quad} & \overline{\mathcal{M}}_{\mathcal{Y}(G, [K])} \end{array}$$

The cells $\overline{\mathcal{M}}_{(G, [K])}$, for graded nodal graphs, also carry canonical orientations.

Definition 5.47. We define orientations for $\overline{\mathcal{M}}_{(G, [K])}(\mathbf{p})$, $(G, [K]) \in \mathcal{SR}_{g,k,l}^m$ by

$$\bar{\mathbf{o}}_{(G, [K])} = \prod_{C \in C(G, [K])} \bar{\mathbf{o}}_C, \quad \mathbf{o}_{(G, [K])} = \bigwedge_{i \in [l]} dp_i \wedge \bar{\mathbf{o}}_{(G, [K])} = \bigwedge_{i \in [l]} \bigwedge_{K(h)=1, h/s_2=i} d\ell_h,$$

the wedge over half edges of face i is taken counterclockwise.

Observation 5.48. Let $(G, [K]) \in \mathcal{SR}_{g,k,l}^m$, $e \in \text{Br}^{eff}(G)$, $(G', [K']) = \mathcal{B}_e(G, [K]) \in \mathcal{SR}_{g,k,l}^{m+1}$, and let e' be the unique edge in G' with two \mathcal{B}_e -preimages. There is a canonical identification

$$\partial_e \overline{\mathcal{M}}_{(G, [K])} \simeq \overline{\mathcal{M}}_{\partial_e(G, [K])} \simeq \mathcal{F}_{e'}, \quad \partial_e \overline{\mathcal{M}}_{(G, [K])}(\mathbf{p}) \simeq \overline{\mathcal{M}}_{\partial_e(G, [K])}(\mathbf{p}) \simeq \mathcal{F}_{e'}(\mathbf{p}),$$

where the space $\mathcal{F}_{e'} \rightarrow \mathcal{M}_{(G', [K'])}$ is the set of pairs (ℓ, x) where $\ell \in \overline{\mathcal{M}}_{(G', [K'])}$, x is a point on e' , with the natural topology. Moreover, the orientation on $\partial_e \overline{\mathcal{M}}_{(G, [K])}(\mathbf{p})$ as an outward boundary of $\overline{\mathcal{M}}_{(G, [K])}(\mathbf{p})$ coincides with the orientation $dx \wedge \mathbf{o}_{(G', [K'])}$, on $\mathcal{F}_{e'}$, where dx is the orientation on the segment e' , considered as a segment in the boundary.

Proof. The only part which requires an explanation is the statement regarding orientations. Recall that K' satisfies $K(h) = K'(\mathcal{B}h)$ for any $h/s_1 \neq e$. It is enough to compare orientations of $\partial_e \overline{\mathcal{M}}_{(G, [K])} \simeq \mathcal{F}_{e'} G'$. Suppose h is the legal side of e , that is, the half edge which satisfies $h/s_1 = e$, $K(h) = 1$. Write $e_{-1} = (s_2^{-1}h)/s_1$, $e_1 = (s_2 h)/s_1$. Then, by recalling the definition of the canonical orientation, Section 5.2, we see that the orientation for $\overline{\mathcal{M}}_{(G, [K])}$ can be written as $d\ell_{e_{-1}} \wedge d\ell_e \wedge d\ell_{e_1} \wedge O$, and the orientation on $\overline{\mathcal{M}}_{G'}$ is $d\ell_{e'} \wedge O$, where O is the wedge of other edge lengths, in some order. Note that $d\ell_{e'} = d\ell_{e_{-1}} + d\ell_{e_1}$. Now, $\partial_e \overline{\mathcal{M}}_{(G, [K])}$, as an outward boundary, is oriented as $d\ell_{e_{-1}} \wedge d\ell_{e_1} \wedge O$. By considering $\mathcal{F}_{e'} G'$ as the moduli of metrics on the graph obtained from G' by adding a new marked point on e' , and with the definition of its

orientation, we see that this orientation can be written as $d\ell_{e_{-1}} \wedge d\ell_{e'} \wedge O$, where $d\ell_{e_{-1}}$ comes from the location of the new point on f . And indeed,

$$d\ell_{e_{-1}} \wedge d\ell_{e_1} \wedge O = d\ell_{e_{-1}} \wedge d\ell_{e'} \wedge O.$$

□

Corollary 5.49. *The map $\text{comb} : \overline{\mathcal{M}}_{g,k,l} \rightarrow \overline{\mathcal{M}}_{g,k,l}^{\text{comb}}$ preserves orientation.*

Proof. Indeed, by Observation 5.48, we see that the orientations on $\overline{\mathcal{M}}_{g,k,l}^{\text{comb}}$ satisfy the same requirements of Lemma 2.57. The dimension 0 case can be checked by hand. □

We also have the following corollary of Corollary 5.35

Corollary 5.50. *For $(G, [K]) \in \mathcal{SR}_{g,k,l}^m$ and an internal edge e which is not a bridge, the orientations on $\partial_e \overline{\mathcal{M}}_{(G,[K])}(\mathbf{p}) \simeq \partial_e \overline{\mathcal{M}}_{(G_e,[K_e])}(\mathbf{p})$, induced as boundaries of $\mathcal{M}_{(G,[K])}(\mathbf{p}), \mathcal{M}_{(G_e,[K_e])}(\mathbf{p})$ are opposite.*

5.3.4. *Canonical sections and intersection numbers.* By the constructions we immediately get

Observation 5.51. (a) For $(G, [K]) \in \mathcal{SR}_{g,k,l}^m, e \in E(G) \setminus \text{Br}(G)$, there is a canonical identification

$$(\mathcal{F}_i \rightarrow \overline{\mathcal{M}}_{\partial_e(G,[K])}) \simeq (\mathcal{F}_i \rightarrow \partial_e \overline{\mathcal{M}}_{(G,[K])}) \simeq (\mathcal{F}_i \rightarrow \partial_e \overline{\mathcal{M}}_{(G_e,[K_e])}),$$

and similarly for the bundles S_L .

(b) For $(G, [K]) \in \mathcal{SR}_{g,k,l}^m, e \in \text{Br}^{\text{eff}}$, then there is a canonical identification

$$(\mathcal{F}_i \rightarrow \overline{\mathcal{M}}_{\partial_e(G,[K])}) \simeq (\mathcal{F}_i \rightarrow \partial_e \overline{\mathcal{M}}_{(G,[K])}) \simeq \mathcal{B}^*(\mathcal{F}_i \rightarrow \mathcal{B}_e \overline{\mathcal{M}}_{(G,[K])}),$$

and similarly for the bundles S_L .

(c) For all i there are canonical identifications

$$\mathcal{F}_i \rightarrow \overline{\mathcal{M}}_{(G,[K])} \simeq \mathcal{Y}^*(\mathcal{F}_i \rightarrow \overline{\mathcal{M}}_{\mathcal{Y}(G,[K])}). \text{ Similarly for } S_L.$$

The identifications are compatible in the sense of Diagram 5.46.

Proposition 5.52. *Let $(s^{(G,[K])})_{(G,[K]) \in \mathcal{SR}_{g,k,l}^0}$ be a special canonical multisection of S_L . Then it induces multisections $(s^{(G,[K])})_{(G,[K]) \in \mathcal{SR}_{g,k,l}^{m \geq 1}}$ which satisfy the condition that for any $(G, [K]) \in \mathcal{SR}_{g,k,l}^m, e \in \text{Br}^{\text{eff}}(G)$,*

$$s^{(G,[K])}|_{\partial_e \overline{\mathcal{M}}_{(G,[K])}} = \mathcal{B}^* s^{\mathcal{B}_e(G,[K])}.$$

In particular, whenever $\partial_{e_1}(G_1, [K_1]) = \partial_{e_2}(G_2, [K_2])$,

$$s^{(G_1,[K_1])}|_{\partial_{e_1} \overline{\mathcal{M}}_{(G_1,[K_1])}} = s^{(G_2,[K_2])}|_{\partial_{e_2} \overline{\mathcal{M}}_{(G_2,[K_2])}},$$

where we compare multisections using the identifications of Observation 5.51.

Proof. Let s be a canonical multisection. Consider $(G, [K]) \in \mathcal{SR}_{g,k,l}^{m \geq 1}$. By Observation 5.42, $(G, [K])$ can be written as $\mathcal{Y}(G', [K'])$, for some trivalent boundary graph. Now $s^{\mathcal{Y}(G', [K'])} = \tilde{\mathcal{B}}^* s^{\tilde{\mathcal{B}}\mathcal{Y}(G', [K'])}$. We have a factorization

$$\begin{array}{ccc} \mathcal{M}_{(G', [K'])} & \xrightarrow{\mathcal{Y}} & \mathcal{M}_{(G, [K])} \\ & \searrow \tilde{\mathcal{B}} & \downarrow \\ & & \mathcal{M}_{\tilde{\mathcal{B}}(G', [K'])}, \end{array}$$

the vertical map is the evident forgetful map, with a finite fiber. The identifications of bundles S_L , see Observations 4.38, 5.51, is also compatible with this diagram. Define $s^{(G, [K])}$ as the pull-back of $s^{\tilde{\mathcal{B}}\mathcal{Y}(G', [K'])}$ along the vertical map. Observe that $s^{\mathcal{Y}(G', [K'])} = \mathcal{Y}^* s^{(G, [K])}$. The required property now follows from Diagram 5.46. \square

6. THE COMBINATORIAL FORMULA

Throughout this sections we shall work with the orientations constructed in Subsection 5.2. These are the same orientations as the ones constructed in [21], by Corollary 5.49.

Definition 6.1. For $(G, [K]) \in \mathcal{SR}_{g,k,l}^m$ define

$$W_G, \widetilde{W}_G : \mathcal{M}_{(G, [K])} \rightarrow \mathbb{R},$$

by

$$W_G(\ell) = \prod_{e \in s_1 H^B(G)} \frac{\ell_e^{2m(e)}}{(m(e) + 1)!}, \quad \widetilde{W}_G(\ell) = \prod_{e \in s_1 H^B(G)} \frac{\ell_e^{2m(e)}}{m(e)!(m(e) + 1)!}.$$

Notation 6.2. Write for $G \in \mathcal{SR}_{g,k,l}^m$ write $\dim(G) = \dim \mathcal{M}_G = 3g - 3 + k + 2l - 2m$.

6.1. Iterative integration and the integral form of the combinatorial formula.

Definition 6.3. A *decoration* D of a graph $(G, [K]) \in \mathcal{SR}_{g,k,l}^m$, is a choice of sets $D_h \subseteq [d]$, for any $h \in s_1 H^B$ which are pairwise disjoint and such that

$$|D_h| = m(h).$$

When $e = h/s_1$ we also write $D_e = D_h$. For a l -set L , a L -decoration is a decoration for which

$$D_h \subseteq L_{i(h)}.$$

In the next series of claims we shall omit $[K]$ from the notation of graded graphs, to make notations shorter.

Denote the collection of all decorations of G by $Dec(G)$, and the collection of all L -decorations of G by $Dec(G, L)$.

Write $L(D) = \bigcup_{h \in s_1 H^B} D_h$, thought as the l -subset of L , defined by $L(D)_i = \bigcup_{i(h)=i} D_h$.

For $(G, [K]) \in \mathcal{SR}_{g,k,l}^{m>0}$ and a (G, L) -decoration D , define the set

$$\mathcal{B}^{-1}(G, D) \subseteq \{(G', e', D') | (G', e') \in \mathcal{B}^{-1}G, D' \in Dec(G', L)\},$$

as follows. $(G', e', D') \in \mathcal{B}^{-1}(G, D)$ exactly when $(G', e') \in \mathcal{B}^{-1}G$, $D' \in Dec(G', L)$, and for any $e \in E(G') \setminus \{e'\}$, $D'_e \subseteq D_{\mathcal{B}e}$. Note that in this case $L(D') \subseteq L(D)$, and the difference is exactly one element.

In order to be able to calculate intersection numbers, we must understand the restriction of the forms α_i, ω_i to the boundary.

Suppose $(G, [K]) \in \mathcal{SR}_{g,k,l}^m$, $e \in Br^{eff}(G)$, h is its illegal side, $K(h) = 1$, and $i \in [l]$. On $\mathcal{M}_{\partial_e G}(\mathbf{p})$ we have two natural representatives for the angular 1-form, $\alpha_i^{\partial_e G} = \alpha_i^G|_{\partial_e \mathcal{M}_G}$, and $\mathcal{B}^* \alpha_i^{\mathcal{B}_e G}$. Similarly, we have two natural choices for the induced 2-forms, $\omega_i^{\partial_e G} = \omega_i^G|_{\partial_e \mathcal{M}_G}$, and $\mathcal{B}^* \omega_i^{\mathcal{B}_e G}$.

Notation 6.4. Write $\beta_i = \beta_i^{\partial_e G} = \alpha_i^{\partial_e G} - \mathcal{B}^* \alpha_i^{\mathcal{B}_e G}$, and $B_i = B_i^{\partial_e G} = \omega_i^{\partial_e G} - \mathcal{B}^* \omega_i^{\mathcal{B}_e G}$.

Observation 6.5. With the above notations, if $i \neq i(e)$, then $B_i = \beta_i = 0$. Otherwise we have

$$p_i^2 \beta_i = \ell_{s_2 h} d\ell_{s_2^{-1} h}, \quad p_i^2 B_i = d\ell_{s_2^{-1} h} \wedge d\ell_{s_2 h}.$$

Proof. For $i \neq i(h)$, the forms restricted from \mathcal{M}_G and those pulled back from the base are canonically identified. Suppose $i = i(h)$, we handle B_i . The proof for β_i is similar. $\ell_e = 0$, hence also $d\ell_e = 0$ on $\partial_e \mathcal{M}_G$. Hence the only difference between $\omega_i^{\partial_e G}$, and $\mathcal{B}^* \omega_i^{\mathcal{B}_e G}$ is that the former may contains terms with $d\ell_{s_2 h}$ or $d\ell_{s_2^{-1} h}$, while the latter depends only on their sum, by the definition of \mathcal{B}_e . Choose a good ordering n in the sense of Definition 5.1, such that half edges of the i^{th} face appear first, and some half edge $h' \neq h, s_2 h$ is the first edge in the ordering. One can always find such a half edge. Otherwise, the i^{th} face is bounded by exactly two edges, $h, s_2 h$, which therefore must be a boundary half edge, and in particular $K(s_2 h) = 1$. But then the sum of K on the i^{th} face is even, which is impossible for a Kasteleyn orientation.

In $\mathcal{B}_e G$ we choose a good ordering n' for which h' , identified as an edge of $\mathcal{B}_e G$, is the first half edge. Suppose $s_2^{-1} h$ is the j^{th} half edge in n , so that $h, s_2 h$ are the $j + 1^{th}, j + 2^{th}$ edges. Write ℓ_a for $\ell_{n^{-1}(a)}$.

Then,

$$\begin{aligned}
p_i^2 \omega_i^G|_{\partial_e \mathcal{M}_G} &= \sum_{a < b} d\ell_a \wedge d\ell_b \\
&= \sum_{a < b, a, b \neq j, j+1, j+2} d\ell_a \wedge d\ell_b + \sum_{a < j} d\ell_a \wedge (d\ell_j + d\ell_{j+2}) + \\
&\quad + \sum_{j+2 < a} (d\ell_j + d\ell_{j+2}) \wedge d\ell_a + d\ell_j \wedge d\ell_{j+2} \\
&= p_i^2 \mathcal{B}^* \omega_i^{\mathcal{B}_e G} + d\ell_j \wedge d\ell_{j+2}.
\end{aligned}$$

In the last equality we used the fact that $\ell_{n'-1(j)}^{\mathcal{B}_e G} = \ell_{n-1(j)} + \ell_{n-1(j+2)}$, and for $a \neq j$, $\ell_{n'-1(a)}^{\mathcal{B}_e G} = \ell_{e_{a+w(a)}}$, where $w(a) = 0$, for $a < j$, and otherwise it is 2. \square

Notation 6.6. Recall Remark 3.5. For G, e as above, given a l -set L , and $i \in L$, we define the form Φ_L^i on the sphere bundle $S_L \rightarrow \partial_e \mathcal{M}_G$

$$\Phi_L^i = \Phi(\{r_j\}_{j \in L}, \{\alpha'_j\}_{j \in L}, \{\omega'_j\}_{j \in L}) = \Phi^{\partial_e G}(\{r_j\}_{j \in L}, \{\alpha'_j\}_{j \in L}, \{\omega'_j\}_{j \in L}),$$

Where $\alpha'_j = \mathcal{B}^* \alpha_j^{\mathcal{B}_e G}$ for $j \neq i$, and $\alpha'_i = \beta_i$. Similarly, $\omega'_j = \mathcal{B}^* \omega_j^{\mathcal{B}_e G}$, unless $j = i$, and then $\omega'_i = B_i$. As usual $\bar{\Phi}_L^i = p^{2L} \Phi_L^i$

From now until the end of this subsection, we fix a l -set L , and let E_L be the corresponding bundle.

Lemma 6.7. *Let s be a special canonical multisection of E_L . Take $G \in \mathcal{SR}_{g,k,l}^m$ arbitrary, e an effective bridge of G , h its illegal side. Let D' be a L -decoration of $\partial_e G$, and write $L' = L(D')$. Then*

$$\int_{\partial_e \mathcal{M}_G(\mathbf{p})} s^*(W_G \bar{\Phi}_{L \setminus L'}) = \sum_{j \in (L \setminus L')_{i(h)}} \int_{\partial_e \mathcal{M}_G(\mathbf{p})} W_G s^*(\bar{\Phi}_{L \setminus L'}^j).$$

Proof. First, the function W_G depends on no variables of the fiber of the sphere bundle, hence can be taken out of the pull-back. Apply Observation 6.5 to $\bar{\Phi}_{L \setminus L'}$, and expand multilinearly. Write $i = i(h)$. Any term containing one or more β_i or B_i will vanish, as a consequence of a multiple appearance of $d\ell_{s_2^{-1}h}$. $s|_{\partial_e \mathcal{M}_G}$ is pulled back from $\mathcal{M}_{\mathcal{B}_e G}$, by Proposition 5.52. Now, a term with no B_i or β_i is a form of degree $\dim_{\mathbb{R}} \mathcal{M}_{\mathcal{B}_e G} + 1$, pulled back from the space $s(\mathcal{M}_{\mathcal{B}_e G})$. Terms of this type vanish because of dimensional reasons. We are left with terms containing a single β_i or B_i . Regrouping we obtain the claim. \square

The second main lemma we need is the following.

Lemma 6.8. Fix $m > 0$, $G \in \mathcal{SR}_{g,k,l}^m$, $D \in \text{Dec}(G, L)$, with $L' = L(D)$. Then

$$\begin{aligned} \sum_{(G', e', D') \in \mathcal{B}^{-1}(G, D)} \int_{\mathcal{M}_{\partial_{e'} G'}(\mathbf{p})} W_{G'} s^* (\bar{\Phi}^{\partial_{e'} G'})_{L \setminus L(D')}^{L' \setminus L(D')} &= \\ &= \int_{\mathcal{M}_G(\mathbf{p})} W_G \bar{\omega}_{L \setminus L'} + \int_{\partial \mathcal{M}_G(\mathbf{p})} W_G s^* (\bar{\Phi}^G)_{L \setminus L'}. \end{aligned}$$

Remark 6.9. Note that $\int_{\mathcal{M}_G(\mathbf{p})} W_G \bar{\omega}_{L \setminus L'}$ does not depend on the multi-section s .

Proof. For convenience we work in the case $|\text{Aut}(G)| = 1$, the general case is handled exactly in the same way, but notations become more complicated. Put

$$E' = \{e \in E(G) \mid m(e) > 0\}.$$

Recall Notation 5.43. Suppose $(G', e') \in \mathcal{B}^{-1}G$ is $\mathcal{B}_{e, a+1}^{-1}G$ for $e \in E'$, $a+1 \in [m(e)]$. Fix $h \in D_e$, and let

$$D(G', h) := \{D' \mid (G', D') \in \mathcal{B}^{-1}(G, D), h \notin L(D')\}.$$

Note that $|D(G', h)| = \binom{m(e)-1}{a}$. Let $e_1 = s_2^{-1}e'$, $e_2 = s_2e'$, be the two half edges of G' mapped under $\mathcal{B}_{e'}$ to e . Then $m(e_1) = a$. Put $\ell'_e = \ell_{e_1}$. For fixed G', h the expression

$$\int_{\mathcal{M}_{\partial_{e'} G'}(\mathbf{p})} W_{G'} s^* \bar{\Phi}_{L \setminus L(D')}^{L' \setminus L(D')} = \int_{\mathcal{M}_{\partial_{e'} G'}} W_{G'} s^* \bar{\Phi}_{L \setminus L(D')}^h$$

hence independent of h . Thus, using Observation 5.48, their sum is

$$\begin{aligned} (29) \quad & \sum_{D' \in D(G', h)} \int_{\mathcal{M}_{\partial_{e'} G'}} W_{G'} s^* \bar{\Phi}_{L \setminus L(D')}^h = \\ &= \int_{\mathcal{M}_G} \binom{m(e)-1}{a} \left(\prod_{f \in E' \setminus \{e\}} \frac{\ell_f^{2m(f)}}{(m(f)+1)!} \right) \cdot \\ & \quad \cdot \int_0^{\ell_e} \frac{(\ell'_e)^{2a} (\ell_e - \ell'_e)^{2(m(e)-a-1)}}{(a+1)!(m(e)-a)!} (A + B + C), \end{aligned}$$

where

$$\begin{aligned}
A &= r_h^2(\ell_e - \ell'_e)d\ell'_e \sum_{n \geq 0} 2^n n! \sum_{|I|=n, I \subseteq L \setminus L'} \left(\bigwedge_{j \in I} r_j dr_j \wedge \bar{\alpha}_j \right) \wedge \bigwedge_{j \in L \setminus (I \cup L')} \bar{\omega}_{L(j)}, \\
B &= r_h dr_h \wedge (\ell_e - \ell'_e)d\ell'_e \sum_{i \in L \setminus L'} r_i^2 \bar{\alpha}_i \sum_{n \geq 0} 2^{(n+1)}(n+1)! \wedge \\
&\quad \wedge \sum_{|I|=n, I \subseteq L \setminus (L' \cup \{i\})} \left(\bigwedge_{j \in I} r_j dr_j \bar{\alpha}_j \right) \\
&\quad \wedge \bigwedge_{j \in L \setminus (L' \cup I \cup \{i\})} \bar{\omega}_{L(j)}, \\
C &= d\ell'_e \wedge d\ell_e \sum_{i \in L \setminus L'} r_i^2 \bar{\alpha}_i \sum_{n \geq 0} 2^n n! \wedge \sum_{|I|=n, I \subseteq L \setminus (L' \cup \{i\})} \left(\bigwedge_{j \in I} r_j dr_j \bar{\alpha}_j \right) \\
&\quad \wedge \bigwedge_{j \in L \setminus (L' \cup I \cup \{i\})} \bar{\omega}_{L(j)},
\end{aligned}$$

$r_h, \bar{\alpha}_j$ and the other variables are evaluated on the section s , which is omitted from notation. We shall use the following proposition.

Proposition 6.10. (a) $\sum_{a=0}^{m-1} \binom{m-1}{a} \int_0^y \frac{x^{2a}(y-x)^{2(m-a)-1}}{(a+1)!(m-a)!} dx = \frac{y^{2m}}{(m+1)!}$
(b) $\sum_{a=0}^{m-1} \binom{m-1}{a} \int_0^y \frac{x^{2a}(y-x)^{2(m-a)-1}}{(a+1)!(m-a)!} dx = \frac{2y^{2m-1}}{(m+1)!}$

Still fixing e , $h \in D_e$, summing Equation 29 over $(G'_a, e'_a) := \mathcal{B}_{e,a+1}^{-1}G$, where $a = 0, \dots, m(e) - 1$, we get, using Proposition 6.10,

$$\begin{aligned}
(30) \quad & \sum_{a=0}^{m(e)-1} \sum_{D' \in D(G'_a, h)} \int_{\mathcal{M}_{\partial_{e'} G'}} W_{G'} s^* \Phi_{L \setminus L(D')}^h = \\
& \int_{\mathcal{M}_G} \prod_{f \in E' \setminus \{e\}} \frac{\ell_f^{2m(f)}}{(m(f) + 1)!} \left\{ \frac{\ell_e^{2m(e)}}{(m(e) + 1)!} r_h^2 \sum_{m \geq 0} 2^m m! \right. \\
& \wedge \sum_{|I|=m, I \subseteq L \setminus L'} \left(\bigwedge_{j \in I} r_j dr_j \wedge \bar{\alpha}_j \right) \wedge \bigwedge_{j \in L \setminus (I \cup L')} \bar{\omega}_{L(j)} \\
& - \frac{\ell_e^{2m(e)}}{(m(e) + 1)!} r_h dr_h \sum_{i \in L \setminus L'} r_i^2 \bar{\alpha}_i \sum_{m \geq 0} 2^{(m+1)} (m+1)! \\
& \wedge \sum_{|I|=m, I \subseteq L \setminus (L' \cup \{i\})} \left(\bigwedge_{j \in I} r_j dr_j \bar{\alpha}_j \right) \wedge \bigwedge_{j \in L \setminus (L' \cup I \cup \{i\})} \bar{\omega}_{L(j)} \\
& + \sum_{i \in L \setminus L'} r_i^2 \bar{\alpha}_i \sum_{m \geq 0} 2^m m! \frac{2m(e) \ell_e^{2m(e)-1} d\ell_e}{(m(e) + 1)!} \\
& \left. \wedge \sum_{|I|=m, I \subseteq L \setminus (L' \cup \{i\})} \left(\bigwedge_{j \in I} r_j dr_j \bar{\alpha}_j \right) \wedge \bigwedge_{j \in L \setminus (L' \cup I \cup \{i\})} \bar{\omega}_{L(j)} \right\}
\end{aligned}$$

Note that in order to perform the sum we have used the last assertion of Proposition 5.52.

The next step is to eliminate r_h terms, for $h \in L'$. For this recall

$$\sum_{h \in L'} r_h^2 = 1 - \sum_{h \in L \setminus L'} r_h^2, \quad \sum_{h \in L'} r_h dr_h = - \sum_{h \in L \setminus L'} r_h dr_h.$$

Summing Equation 30 over $e \in E'$, $h \in D_e$, and using the last identities we get

$$\begin{aligned}
(31) \quad & \sum_{(G', e', D') \in \mathcal{B}^{-1}(G, D)} \int_{\mathcal{M}_{\partial_{e'} G'}} W_{G'} s^* \bar{\Phi}_{L \setminus L(D')}^{L(D) \setminus L(D')} = \\
& = \int_{\mathcal{M}_G} \left(\prod_{f \in E'} \frac{\ell_f^{2m(f)}}{(m(f) + 1)!} \right) X + \\
& + \int_{\mathcal{M}_G} \left(\sum_{e \in E'} \frac{2m(e) \ell_e^{2m(e)-1} d\ell_e}{(m(e) + 1)!} \prod_{f \in E' \setminus \{e\}} \frac{\ell_f^{2m(f)}}{(m(f) + 1)!} \right) Y,
\end{aligned}$$

where

$$\begin{aligned}
X &= \left(1 - \sum_{h \in L \setminus L'} r_h^2\right) \wedge \\
&\wedge \sum_{m \geq 0} 2^m m! \sum_{|I|=m, I \subseteq L \setminus L'} \left(\bigwedge_{j \in I} r_j dr_j \wedge \bar{\alpha}_j \right) \wedge \bigwedge_{j \in L \setminus (I \cup L')} \bar{\omega}_{L(j)} + \\
&+ \left(\sum_{h \in L \setminus L'} r_h dr_h \right) \sum_{i \in L \setminus (L' \cup \{h\})} r_i^2 \bar{\alpha}_i \wedge \sum_{m \geq 0} 2^{(m+1)} (m+1)! \times \\
&\quad \times \sum_{|I|=m, I \subseteq L \setminus (L' \cup \{i, h\})} \left(\bigwedge_{j \in I} r_j dr_j \wedge \bar{\alpha}_j \right) \wedge \bigwedge_{j \in L \setminus (L' \cup I \cup \{i\})} \bar{\omega}_{L(j)}, \\
Y &= \sum_{i \in L \setminus L'} r_i^2 \bar{\alpha}_i \wedge \\
&\quad \wedge \sum_{m \geq 0} 2^m m! \sum_{|I|=m, I \subseteq L \setminus (L' \cup \{i\})} \left(\bigwedge_{j \in I} r_j dr_j \wedge \bar{\alpha}_j \right) \wedge \\
&\quad \wedge \bigwedge_{j \in L \setminus (L' \cup I \cup \{i\})} \bar{\omega}_{L(j)}.
\end{aligned}$$

A direct calculation, using Stokes' theorem, shows that the right hand side of Equation 31 is exactly

$$\int_{\mathcal{M}_G} \left\{ \prod_{e \in E'} \frac{\ell_e^{2m(e)}}{(m(e)+1)!} \bigwedge_{i \in L \setminus L'} \bar{\omega}_{L(i)} + d \left(\prod_{e \in E'} \frac{\ell_e^{2m(e)}}{(m(e)+1)!} \bar{\Phi}_{L \setminus L'} \right) \right\}.$$

□

Proof of Proposition 6.10. We first prove part (b). Write

$$f(x) = \sum_{m=0}^{\infty} \frac{x^{2m}}{m!(m+1)!}.$$

The identity we need to prove is equivalent to

$$(f * f)(x) = f'(x),$$

where $*$ is the convolution. Using Laplace transform, the last equation is equivalent to

$$F^2(\lambda) = \lambda F(\lambda) - 1,$$

where

$$F(\lambda) = \int_0^\infty e^{-\lambda x} f(x) dx$$

is the Laplace transform of f . Expanding F we obtain

$$(32) \quad F = \sum_{m=0}^{\infty} \frac{1}{m!(m+1)!} \int_0^\infty e^{-\lambda x} x^{2m} = \sum_{m=0}^{\infty} \frac{(2m)!}{m!(m+1)!} \lambda^{-2m-1} = \\ = \lambda^{-1} \frac{1 - \sqrt{1 - 4\lambda^{-2}}}{2\lambda^{-2}} = \lambda \frac{1 - \sqrt{1 - 4\lambda^{-2}}}{2},$$

the third equation is a consequence the general binomial formula. Thus, we are left with verifying that

$$F^2(\lambda) = \frac{\lambda^2}{2} (1 - \sqrt{1 - 4\lambda^{-2}}) - 1 = \lambda F(\lambda) - 1,$$

which is straightforward.

The first identity is a consequence of the second. Indeed, Write

$$I_m = \sum_{a=0}^{m-1} \binom{m-1}{a} \int_0^y \frac{x^{2a}(y-x)^{2(m-a)-1}}{(a+1)!(m-a)!} dx, \\ J_m = \sum_{a=0}^{m-1} \binom{m-1}{a} \int_0^y \frac{x^{2a}(y-x)^{2(m-a-1)}}{(a+1)!(m-a)!} dx.$$

It suffices to show that

$$I_m = \frac{y}{2} J_m.$$

Indeed,

$$(33) \quad I_m = \sum_{a=0}^{m-1} \binom{m-1}{a} \int_0^y \frac{x^{2a}(y-x)^{2(m-a)-1}}{(a+1)!(m-a)!} dx \\ = y \sum_{a=0}^{m-1} \binom{m-1}{a} \int_0^y \frac{x^{2a}(y-x)^{2(m-a-1)}}{(a+1)!(m-a)!} dx - \\ - \sum_{a=0}^{m-1} \binom{m-1}{a} \int_0^y \frac{x^{2a+1}(y-x)^{2(m-a-1)}}{(a+1)!(m-a)!} dx \\ = y J_m - \sum_{a=0}^{m-1} \binom{m-1}{a} \int_0^y \frac{(y-t)^{2a+1} t^{2(m-a-1)}}{(a+1)!(m-a)!} dx \\ = y J_m - I_m,$$

where the second equality follows from opening one $(y-x)$ term, and the third follows from the substitution $t = y - x$. \square

In order to be able to write an expression for the open intersection numbers we need the following observation.

Observation 6.11. Suppose $G \in \mathcal{SR}_{g,k,l}^m$, and e an edge with $m(e) > 0$. Then for any decoration D ,

$$\int_{\partial_e \mathcal{M}_G(\mathbf{p})} W_G s^* \bar{\Phi}_{L \setminus L(D)} = 0$$

Proof. It follows from the definition of W_G that $W_G|_{\mathcal{M}_{\partial_e G}} = 0$ identically. \square

We can now state and prove the integral form of the combinatorial formula.

Theorem 6.12. *Let L be a decoration. Set $a_i = |L_i|$. Then*

$$(34) \quad p^{2L} 2^{\frac{g+k-1}{2}} \langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \rangle = \sum_{G \in \tilde{\mathcal{SR}}_{g,k,l}^*} \sum_{D \in \text{Dec}(G,L)} \int_{\mathcal{M}_G(\mathbf{p})} W_G \bar{\omega}_{L \setminus L(D)}$$

Proof. Using Lemma 4.45, we can write

$$\begin{aligned} p^{2L} 2^{\frac{g+k-1}{2}} \langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \rangle &= \sum_{(G,[K]) \in \mathcal{SR}_{g,k,l}^0} \int_{\mathcal{M}_G(\mathbf{p})} \bar{\omega}_L + \\ &+ \sum_{(G,[K]) \in \mathcal{SR}_{g,k,l}^0} \sum_{[e] \in [\text{Br}^{\text{eff}}(G)]} \int_{\mathcal{M}_{\partial_e(G,[K])}(\mathbf{p})} s^* \bar{\Phi}_L \\ &= A_0 + S_0, \end{aligned}$$

where we define

$$\begin{aligned} A_m &= \sum_{(G,[K]) \in \mathcal{SR}_{g,k,l}^m} \sum_{D \in \text{Dec}(G,L)} \int_{\mathcal{M}_{(G,[K])}(\mathbf{p})} W_G \bar{\omega}_{L \setminus L(D)} \\ S_m &= \sum_{(G,[K]) \in \mathcal{SR}_{g,k,l}^m} \sum_{D \in \text{Dec}(G,L)} \int_{\partial \mathcal{M}(\mathbf{p})_{(G,[K])}} W_G s^* \bar{\Phi}_{L \setminus L(D)}, \end{aligned}$$

and s is nowhere vanishing special canonical.

We now claim

$$S_m = A_{m+1} + S_{m+1}.$$

Indeed, consider S_m . Recall that for any G ,

$$\partial \mathcal{M}_{(G,[K])} = \bigcup_{[e] \in [E(G)]} \partial_e \overline{\mathcal{M}}_{(G,[K])} = \bigcup_{[e] \in [E(G)]} \overline{\mathcal{M}}_{\partial_e(G,[K])}.$$

Since for different edges the boundary cells intersect in low dimension, the integral over the union is just the sum of integrals over $\partial_e \overline{\mathcal{M}}_{(G,[K])}$, over $[E]$. For an edge e which is not a bridge, by Corollary 5.50, we know that $\partial_e \overline{\mathcal{M}}_{(G,[K])}(\mathbf{p}) = -\partial_e \overline{\mathcal{M}}_{(G,[K])_e}(\mathbf{p})$ considered as oriented orbifolds, with the orientation induced as a boundary.

Now, $Dec(G, L), Dec(G_e, L)$ are the same sets, and it is easy to see that

$$W_G|_{\partial_e \overline{\mathcal{M}}_{(G,[K])}} = W_{G_e}|_{\partial_e \overline{\mathcal{M}}_{(G,[K])_e}}.$$

Thus, given a decoration D ,

$$\int_{\partial_e \overline{\mathcal{M}}_{(G,[K])}(\mathbf{p})} W_G s^* \bar{\Phi}_{L \setminus L(D)} = - \int_{\partial_e \overline{\mathcal{M}}_{(G,[K])_e}(\mathbf{p})} W_{G_e} s^* \bar{\Phi}_{L \setminus L(D)}.$$

Suppose now e is a bridge which is not effective. From Observation 6.11, for any decoration D

$$\int_{\partial_e \mathcal{M}_{(G,[K])}(\mathbf{p})} W_G s^* \bar{\Phi}_{L \setminus L(D)} = 0.$$

Thus, we can write,

$$S_m = \sum_{(G,[K]) \in \mathcal{SR}_{g,k,l}^m} \sum_{D \in Dec(G,L)} \sum_{[e] \in [Br^{eff}(G)]} \int_{\mathcal{M}_{\partial_e(G,[K])}(\mathbf{p})} W_G s^* \bar{\Phi}_{L \setminus L(D)}.$$

Applying Lemma 6.7, we obtain

$$S_m = \sum_{(G,[K]) \in \mathcal{SR}_{g,k,l}^m} \sum_{D \in Dec(G,L)} \sum_{[e] \in [Br^{eff}(G)]} \sum_{j \in (L \setminus L(D))_{i(e)}} \int_{\mathcal{M}_{\partial_e(G,[K])}(\mathbf{p})} W_G s^* \bar{\Phi}_{L \setminus L(D)}^j.$$

Note that when e is an effective bridge, then $G' = \mathcal{B}_e(G, [K]) \in \mathcal{SR}_{g,k,l}^{m+1}$. In addition, $j \in (L \setminus L(D))_{i(e)}$ induces a single decoration D' of G' , which is defined by $(G, D) \in \mathcal{B}^{-1}(G', D')$ and $j \in L(D')$. Moreover, any $(G', [K']) \in \mathcal{SR}_{g,k,l}^{m+1}$, $D' \in Dec(G', L)$ is obtained in this way, see Observation 5.42. Hence, we can apply Lemma 6.8 and get

$$\begin{aligned} S_m &= \sum_{(G,[K]) \in \mathcal{SR}_{g,k,l}^{m+1}} \sum_{D \in Dec(G,L)} \int_{\mathcal{M}_{(G,[K])}(\mathbf{p})} W_G \bar{\omega}_{L \setminus L(D)} + \\ &\quad + \sum_{(G,[K]) \in \mathcal{SR}_{g,k,l}^{m+1}} \sum_{D \in Dec(G,L)} \int_{\partial \overline{\mathcal{M}}_{(G,[K])}(\mathbf{p})} W_G s^* \bar{\Phi}_{L \setminus L(D)} \\ &= A_{m+1} + S_{m+1}, \end{aligned}$$

as claimed. Iterating over m , we see that the left hand side of Equation 34 is $\sum_{m \geq 0} A_m$.

We now claim

Proposition 6.13. *If G is a nodal graph such that on at least one boundary component there is an even total number of boundary marked points and legal nodes on, then*

$$\int_{\mathcal{M}_{(G,[K])}(\mathbf{p})} W_G \bar{\omega}_{L \setminus L(D)} = 0.$$

The proof is given in Subsection 6.2, see Lemma 6.21.

Thus,

$$\sum_{m \geq 0} A_m = \sum_{m \geq 0} \sum_{(G,[K]) \in \tilde{\mathcal{R}}_{g,k,l}^m} \sum_{D \in \text{Dec}(G,L)} \int_{\mathcal{M}_{(G,[K])}(\mathbf{p})} W_G \bar{\omega}_{L \setminus L(D)},$$

as claimed. \square

Open problem 2. *The moduli space $\overline{\mathcal{M}}_{g,k,l}$ is disconnected, and is composed of components which parameterize different topologies and graded structures. The boundary conditions of [21, 20] define in fact an intersection number on each such component, and their sum is what we denote by $\langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \rangle_g$. Using the techniques presented in this section one can actually calculate all these smaller intersection numbers. A natural question is whether these numbers also satisfy interesting relations.*

Observation 6.14.

$$\begin{aligned} |\text{Dec}(G, L)| &= \left(\begin{matrix} L_i \\ \{m(e) | e \in E, i(e) = i\} \end{matrix} \right) = \\ &= \prod_{i \in [l]} \frac{L_i!}{\left(\prod_{\{e \in E | i(e) = i\}} m(e)! \right) (L_i - \sum_{\{e \in E | i(e) = i\}} m(e))!}. \end{aligned}$$

Thus, with the above notations we have,

$$\begin{aligned} 2^{\frac{g+k-1}{2}} \prod_{i \in [l]} p_i^{2a_i} \langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \rangle &= \\ \sum_{m \geq 0} \sum_{(G,[K]) \in \tilde{\mathcal{R}}_{g,k,l}^m} \left(\prod_{i \in [l]} \left(\begin{matrix} a_i \\ \{m(e) | e \in E, i(e) = i\} \end{matrix} \right) \right) \int_{\mathcal{M}_{(G,[K])}(\mathbf{p})} W_G \bar{\omega}_{L \setminus L(D)} &= \\ = \sum_{m \geq 0} \sum_{(G,[K]) \in \tilde{\mathcal{R}}_{g,k,l}^m} \left(\prod_{i \in [l]} \frac{a_i!}{(a_i - \sum_{\{e \in E | i(e) = i\}} m(e))!} \right) \int_{\mathcal{M}_{(G,[K])}(\mathbf{p})} \widetilde{W}_G \bar{\omega}_{L \setminus L(D)}, \end{aligned}$$

where \widetilde{W}_G is defined in Definition 6.1, and $D \in D(G, L)$ are arbitrary decorations. Summing over all possible L , and dividing by $d!$, we get

$$(35) \quad 2^{\frac{g+k-1}{2}} \sum_{\sum a_i=d} \prod_{i \in [l]} \frac{p_i^{2a_i}}{a_i!} \langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \rangle =$$

$$\sum_{m \geq 0} \sum_{(G, [K]) \in \tilde{\mathcal{R}}_{g,k,l}^m} \int_{\mathcal{M}_{(G, [K])}(\mathfrak{p})} \widetilde{W}_G \frac{\bar{\omega}^{d-m}}{(d-m)!}$$

Dimensional reasons give,

Observation 6.15. Let L' be a l -set, $(G, [K]) \in \tilde{\mathcal{R}}_{g,k,l}^*$. Suppose that for some component $C \in C(G, [K])$,

$$\dim(C) < \sum_{i \in I(C)} L'_i.$$

then $\int_{\mathcal{M}_G} f \omega_{L'} = 0$, for any function f .

Now, $\bar{\omega} = \sum_{C \in C(G)} \bar{\omega}^C$, where $\bar{\omega}^C = \sum_{i \in I(C)} \bar{\omega}_i$. Thus, together with the observation we get,

Corollary 6.16. $\widetilde{W}_G \frac{\bar{\omega}^{d-m}}{(d-m)!} = \prod_{C \in C(G)} \widetilde{W}_C \frac{(\bar{\omega}^C)^{\dim(C)}}{\dim(C)!}$. Thus,

$$(36) \quad \sum_{\sum a_i=d} \prod_{i \in [l]} \frac{p_i^{2a_i}}{a_i!} 2^{\frac{k-1}{2}} \langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \rangle =$$

$$\sum_{m \geq 0} \sum_{(G, [K]) \in \tilde{\mathcal{R}}_{g,k,l}^m} \int_{\mathcal{M}_{(G, [K])}(\mathfrak{p})} \widetilde{W}_G \prod_{C \in C(G, [K])} \frac{(\bar{\omega}^C)^{\dim(C)}}{\dim(C)!} =$$

$$\sum_{m \geq 0} \sum_{(G, [K]) \in \tilde{\mathcal{R}}_{g,k,l}^m} \prod_{C \in C(G, [K])} \int_{\mathcal{M}_C} \widetilde{W}_C \frac{(\bar{\omega}^C)^{\dim(C)}}{\dim(C)!}$$

6.2. Power of 2. The aim of this subsection is to gain a better understanding of the forms $\bigwedge dp_i \wedge \frac{\bar{\omega}^d}{d!}, \mathfrak{o}_{(G, [K])}$ and their ratio.

Definition 6.17. For $(G, [K]) \in \mathcal{R}_{g,k,l}^*$ define $s(G, [K])$, to be the sign of

$$\bigwedge dp_i \wedge \frac{\bar{\omega}^d}{d!} : \mathfrak{o}_{(G, [K])}.$$

For $G \in \mathcal{R}_{g,k,l}^*$ define

$$c_{spin}(G) = \sum_{[K] \in [K(G)]} s(G, [K]).$$

Lemma 6.18. For $G \in \mathcal{SR}_{g,k,l}^*$,

$$\bigwedge dp_i \wedge \frac{\bar{\omega}^d}{d!} : \mathfrak{o}_{(G,[K])} = s(G, [K]) c_{spin}(G) 2^{|V^I(G)|}.$$

Proof. Both the left hand side and the right hand side are multiplicative with respect to taking non-nodal components, by the first statement in 6.16 and the construction of $\mathfrak{o}_{(G,[K])}$, thus, it is enough to prove the lemma for graphs in $\mathcal{SR}_{g,k,l}^0$.

Recall that $|K(G)| = 2^{V^I}$, by Lemma 5.9. In addition, by Lemma 5.30, $\mathfrak{o}_{(G,K)}$ for different $K \in [K]$ are equal. Thus, Equation 37 is equivalent to the lemma.

$$(37) \quad \bigwedge dp_i \wedge \frac{\bar{\omega}^d}{d!} = \sum_{K \in K(G)} \mathfrak{o}_{(G,[K])}.$$

Recall $\bar{\omega} = \sum_{i=1}^l \bar{\omega}_i$. Fix a good ordering n . In order to prove Equation 37, it will be more comfortable to work with new variables $\ell_h, h \in H^I$, instead of $\ell_e, e \in E$. Set

$$\begin{aligned} H_{K,i} &= \{h \in H_K \mid h/s_2 = i\}, \\ d_{K,i} &= \frac{|H_{K,i}| - 1}{2}, \\ p_{K,i} &= \sum_{h \in H_{K,i}} \ell_h, \\ \bar{\omega}_{K,i} &= \sum_{h_1, h_2 \in H_{K,i}, n(h_1) < n(h_2)} d\ell_{h_1} \wedge d\ell_{h_2}. \end{aligned}$$

Remark 6.19. Note that only $\bar{\omega}_{K,i}$ depends on the ordering n . For different orders the change in $\bar{\omega}_{K,i}$ is of the form $dp_{K,i} \wedge dx$, where x is a linear combination of $\{d\ell_h\}_h \in H_{K,i}$. Thus, for any a the form $dp_{K,i} \wedge \bar{\omega}_{K,i}^a$ is independent of n .

Express each dp_i by $\sum_{h \in H_i} d\ell_h$, and express also each $\bar{\omega}_i$ in the $\{d\ell_h\}_{h \in H^I}$ basis as above. Our next aim is to show that

$$(38) \quad \bigwedge dp_i \wedge \frac{\bar{\omega}^d}{d!} = \sum_{K \in K(G)} \bigwedge_{i \in [l]} dp_{K,i} \wedge \frac{\bar{\omega}_{K,i}^{d_{K,i}}}{d_{K,i}!} (\text{mod } I),$$

where I is the ideal $(d\ell_h - d\ell_{s_1 h})_{h \in H^I}$. In order to show Equation 38 expand $\bigwedge dp_i \wedge \frac{\bar{\omega}^d}{d!}$ multilinearly, in terms of $\{d\ell_h\}_{h \in H^I}$, without cancellations. Any monomial which appears in this expression, and contains exactly one of $d\ell_h, d\ell_{s_1 h}$ for any $h \in H^I$, defines a unique Kasteleyn orientation K , defined by $K(h) = 1$ if and only if $d\ell_h$ appears in the

monomial. This is indeed a Kasteleyn orientation since any $h \in s_1 H^B$ has $K(h) = 1$, and for any $i \in [l]$, an odd number of variables of half edges appear, one comes from dp_i , and the others come in pairs via powers of $\bar{\omega}_i$.

It is transparent that any Kasteleyn orientation $K \in K(G)$, is generated this way. Moreover, regrouping all terms which correspond to the same Kasteleyn orientation, and using the identity

$$\left(\sum_{i=1}^{2m+1} x_i \right) \wedge \frac{(\sum_{i < j} x_i \wedge x_j)^m}{m!} = x_1 \wedge x_2 \wedge \dots \wedge x_{2m+1},$$

we get Equation 38. \square

Proposition 6.20. *For $G \in \mathcal{SR}_{g,k,l}^0$, $e \in Br(G)$,*

$$c_{spin}(G) = c_{spin}(G_e)$$

Proof. It follows from Lemma 6.18 that

$$c_{spin}(G) = \pm \sum_{[K'] \in [K(G)]} \mathfrak{o}_{(G,[K'])} : \mathfrak{o}_{(G,[K])},$$

for any fixed $[K] \in [K(G)]$. If $K, K' \in K(G)$, then by the orientability of the moduli, Theorem 5.33, we see that

$$\mathfrak{o}_{(G,[K])} : \mathfrak{o}_{(G,[K'])} = \mathfrak{o}_{(G,[K_e])} : \mathfrak{o}_{(G,[K'_e])},$$

as $(G, [K])$, $(G, [K_e])$ and $(G, [K'])$, $(G, [K'_e])$ parameterize adjacent cells. Thus, $c_{spin}(G) = \pm c_{spin}(G_e)$. But $c_{spin} \geq 0$, hence the equality. \square

Lemma 6.21. *If $G \in \mathcal{R}_{g,k,l}^m \setminus \tilde{\mathcal{R}}_{g,k,l}^m$, $c_{spin} = 0$.*

Proof. Again, as c_{spin} is multiplicative in non-nodal components, it is enough to consider the case of non nodal graphs. Let $\partial\Sigma_b$ be a boundary with an even number of boundary marked point. Note that given a surface Σ , and a boundary component $\partial\Sigma_b$, graded spin structures on Σ can be partitioned into pairs which differ exactly in the lifting of $\partial\Sigma_b$. Thus, we can partition $[K(G)]$ into pairs which differ exactly in the boundary conditions at $\partial\Sigma_b$. In combinatorial terms, for any pair $\{(G, [K_1]), (G, [K_2])\}$ in the partition we can find $K_1 \in [K_1]$, $K_2 \in [K_2]$ which agree everywhere, except on edges with exactly one vertex in $\partial\Sigma_b$, where they disagree. We shall show that $s(G, [K_1]) = -s(G, [K_2])$.

As a consequence of the Proposition 6.20 $c_{spin}(G, [K]) = c_{spin}(G_e, [K_e])$, $G \in \mathcal{R}_{g,k,l}^0$, $e \in E(G) \setminus Br(G)$. By performing enough such Feynman moves at boundary edges of G , see Figure 3, moves b, c , we may assume only one non-boundary edge emanates from $\partial\Sigma_b$. Let $2a$ denote the number of the boundary marked points on $\partial\Sigma_b$. Note that $\partial\Sigma_b$ is

part of the boundary of a single face, say face 1. Let $h, s_1(h)$ be the internal half edges which touch $\partial\Sigma_b$. Choose a good ordering n on G , so that $n(h) = 1, n(h_1) = 2, \dots, n(h_{2a+1}) = 2a + 2, n(s_1h) = 2a + 3$ where $h_i \in H^I$ are the other half edges on $\partial\Sigma_b$. This can always be done, possibly after interchanging h and s_1h . Choose any $K_1 \in [K_1]$, and $K_2 \in [K_2]$, which differ only in their values at h, s_1h . Thus, the sign difference between $\mathfrak{o}_{(G, [K_1])}$ and $\mathfrak{o}_{(G, [K_2])}$ is just $(-1)^{2a+1} = -1$, since we change only the location of the variable $d\ell_{h/s_1}$, by $2a + 1$ spots. As claimed. \square

We can now prove Proposition 6.13

Proof. By Lemma 6.18 the proposition is equivalent to $c_{spin}(G) = 0$. But $c_{spin}(G) = \prod_{C \in C(G)} c_{spin}(C)$, which is 0 by Lemma 6.21. \square

Lemma 6.22. *For $G \in \tilde{\mathcal{R}}_{g,k,l}^0$, we have*

$$c_{spin}(G) = 2^{\frac{g+b-1}{2}},$$

where g is the genus of G , and b , is the number of boundaries. For $G \in \tilde{\mathcal{R}}_{g,k,l}^m$, $c_{spin}(G) = \prod c_{spin}(G_i)$, where G_i are the smooth components of G .

Proof. Again it is enough to consider non-nodal graphs. By Equation 37, and the fact $\mathfrak{o}_{(G, [K_1])} = \pm \mathfrak{o}_{(G, [K_2])}$, for any $K_1, K_2 \in [K(G)]$, we see that $c_{spin}(G) \geq 0$. As a consequence of Proposition 6.20 $c_{spin}(G, [K]) = c_{spin}(G_e, [K_e])$, whenever $G \in \tilde{\mathcal{S}}\mathcal{R}_{g,k,l}^0, e \in E(G) \setminus Br(G)$. Thus, it is enough to calculate c_{spin} for the graph \bar{G} , where G is the graph constructed in Example 5.19, see Figure 4. We shall work with the notation of that example. We shall order the faces according to their labels, and we choose an ordering n of the edges of face 1 such that a_1 is the first edge. Choose a Kasteleyn orientation and write

$$\begin{aligned} \mathfrak{o}_G = W_1 \wedge W_2 \wedge \dots \wedge W_{g_s} \wedge d\ell_{h_2} \wedge d\ell_{x_2} \dots d\ell_{h_l} \wedge d\ell_{x_l} \wedge d \\ \wedge \ell_{e_{10}} \dots \wedge d\ell_{e_{1k_1}} \wedge R \wedge d\ell_{y_2} \dots d\ell_{y_l}, \end{aligned}$$

where W_i is the wedge of $d\ell_{a_i}, d\ell_{b_i}, d\ell_{c_i}, d\ell_{d_i}, d\ell_{f_i}, d\ell_{g_i}$, according to the order induced by K , R is the wedge of the remaining variables, according to the ordering. The ordering n , restricted to the half edges which are involved in W_i , is

$$a_i, f_i, d_i, \bar{g}_i, c_i, \bar{f}_i, b_i, g_i.$$

There are four possibilities for $K(\bar{f}_i), K(\bar{g}_i)$. Let K_i^0 denote the set of possibilities with $K(\bar{f}_i)K(\bar{g}_i) = 0$. Let K_i^1 be the singleton made of the remaining possibility. One can check by hand that the form W_i is constant in K_i^0 , and minus that constant in the forth possibility.

The ordering restricted to the remaining edges is

$$b_{12}, e_{2(k_2+1)}, b_{23}, e_{3(k_3+1)} \dots, b_{(b-1)b}, e_{b0}, e_{b1}, \dots, e_{bk_b}, \\ \bar{b}_{(b-1)b}, e_{(b-1)0}, e_{(b-1)1}, \dots, e_{(b-1)k_{-1b}}, \bar{b}_{(b-2)(b-1)}, e_{(b-2)0} \dots, e_{2k_2} \bar{b}_{12}.$$

The only freedom in K is in the values of $K(b_{j(j+1)})$. The relative order of these edges is

$$b_{12}, b_{23}, \dots, b_{(b-1)b}, \bar{b}_{(b-1)b}, \dots, \bar{b}_{12}.$$

Observe that between $b_{j(j+1)}$ and $\bar{b}_{j(j+1)}$ in the ordering, there is an even number of half edges. Thus, different assignments of $K(b_{j(j+1)})$ do not change the orientation \mathfrak{o}_G . There are 2^{b-1} such assignments, where b is the number of boundary components.

To summarize, $s(G, [K])$ depends only on $\sum_i K(\bar{f}_i)K(\bar{g}_i)$, which is just the parity of the graded spin structure, see Remark 5.20, and different parities give rise to different signs. By the calculation in Remark 5.20 we see that $c_{spin}(G) = \pm 2^{\frac{g-b+1}{2}+b-1}$, but since it cannot be negative we end with $c_{spin}(G) = 2^{\frac{g+b-1}{2}}$. \square

Remark 6.23. An analogous power of 2 appears in [15] when one wants to calculate the Laplace transform of the integral combinatorial formula. The method developed in this paper is also applicable to that calculation. It shows exactly where this power of 2 comes from, and how is it connected to spin structures. In fact, our c_{spin} can be thought as an open analog of Witten's class for $r = 2$ -spin, see [25].

Corollary 6.24. *For $G \in \mathcal{SR}_{g,k,l}^0$,*

$$\bigwedge dp_i \wedge \frac{\bar{\omega}^d}{d!} : \mathfrak{o}_{(G,[K])} = s(G, [K]) 2^{|V^I(G)| + \frac{g(G)+b(G)-1}{2}}.$$

6.3. Laplace transform and the combinatorial formula. As in the closed case, a more compact formula may be obtained after performing a Laplace transform to 6.16.

Let λ_i be the variable dual to p_i and write, for $e = \{h_1, h_2 = s_1 h_1\}$,

$$\lambda(e) = \begin{cases} \frac{1}{\lambda_i + \lambda_j} & i(h_1) = i, i(h_2) = j \\ \frac{1}{m(e)+1} \binom{2m(e)}{m(e)} \lambda_i^{-2m(e)-1} & i(h_1) = i, h_2 \in H^B. \end{cases}$$

We also define $\tilde{\lambda}(e) = \frac{1}{\lambda(e)}$ for an internal edge and $\tilde{\lambda}(e) = \lambda_{i(e)}$ for a boundary edge of face i .

Applying the transform to the left hand side of 6.16 gives

$$\int_{p_1, \dots, p_l > 0} \bigwedge dp_i e^{-\sum \lambda_i p_i} \sum_{\sum a_i = d} \prod_{i \in [l]} \frac{p_i^{2a_i}}{a_i!} 2^{\frac{g+k-1}{2}} \langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \rangle =$$

$$2^{d + \frac{g+k-1}{2}} \sum_{\sum a_i = d} \prod_{i \in [l]} \frac{(2a_i - 1)!!}{\lambda_i^{2a_i+1}} \langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \rangle,$$

where $d = \frac{k+2l+3g-3}{2}$.

Transforming the right hand side leaves us with

$$\sum_{m \geq 0} \sum_{G \in \tilde{\mathcal{R}}_{g,k,l}^m} \int_{p_1, \dots, p_l > 0} \bigwedge dp_i e^{-\sum \lambda_i p_i} \prod_{C \in C(G, [K])} \int_{\mathcal{M}_C} \widetilde{W}_C \frac{(\bar{\omega}^C)^{\dim(C)}}{\dim(C)!} =$$

$$\sum_{m \geq 0} \sum_{G \in \tilde{\mathcal{R}}_{g,k,l}^m} \int_{p_1, \dots, p_l > 0} \bigwedge dp_i e^{-\sum \tilde{\lambda}(e) \ell_e} \prod_{C \in C(G, [K])} \int_{\mathcal{M}_C} \widetilde{W}_C \frac{(\bar{\omega}^C)^{\dim(C)}}{\dim(C)!}$$

where we have used the fact that the perimeter of a face is the sum of its edges' lengths.

Recall that

$$\prod_{C \in C(G, [K])} \widetilde{W}_C = \prod_{e \in E^B(G)} \frac{\ell_e^{2m(e)}}{(m(e))!(m(e) + 1)!}.$$

By Corollary 6.24, applied to $(G, [K]) \in \tilde{\mathcal{R}}_{g,k,l}^0$, we have

$$\frac{\left(\bigwedge_{i \in [l]} dp_i \right) \bar{\omega}^d / d!}{\bigwedge_{e \in E(G)} d\ell_e} = s(G, [K]) 2^{|V^I(G)| + \frac{g(G)+b(G)-1}{2}},$$

the variables in the denominator are ordered by $\mathfrak{o}_{(G, [K])}$, and $|V^I|, g, b$ are the number of internal vertices of G , its genus and the number of boundary components, respectively. In addition, $\sum_{[K] \in [K(G)]} s(G, [K]) = c_{spin} = 2^{\frac{g+b-1}{2}}$, by Lemma 6.22. Moreover, since $Aut(G)$ acts on $[K(G)]$, and is sign preserving, we see that

$$\sum_{[K] \in [K(G)]} s(G, [K]) / |Aut(G)| = \sum_{[K] \in [K(G)] / Aut(G)} s(G, [K]) / |Aut(G, [K])|.$$

Thus, for a fixed $G \in \tilde{\mathcal{R}}_{g,k,l}^m$, summing over $for_{spin}^{-1}(G)$ using Observation 5.39, and recalling that $\mathcal{M}_{(G, [K])} \simeq \mathbb{R}^{E(G)} / |Aut(G, [K])|$, we

get

$$\begin{aligned}
& \sum_{[K]} \frac{1}{|Aut(G, [K])|} \int_{p_1, \dots, p_l > 0} \bigwedge dp_i e^{-\sum \tilde{\lambda}(e) \ell_e} \prod_{C \in C(G, [K])} \int_{\mathbb{R}^{E(C)}} \widetilde{W}_C \frac{(\bar{\omega}^C)^{\dim(C)}}{\dim(C)!} = \\
& = \frac{\prod_{C \in C(G)} c(C)}{|Aut(G)|} \prod_{e \in E \setminus E^B} \int_0^\infty e^{-\tilde{\lambda}(e) \ell_e} d\ell_e \prod_{e \in E^B} \int_0^\infty e^{-\tilde{\lambda}(e) \ell_e} \frac{\ell_e^{2m(e)}}{m(e)!(m(e)+1)!} d\ell_e = \\
& = \frac{\prod_{C \in C(G)} c(C)}{|Aut(G)|} \prod_{e \in E} \lambda(e),
\end{aligned}$$

where $c(C) = 2^{|V^I(C)|+g(C)+b(C)-1}$. Summing over all $G \in \tilde{\mathcal{R}}_{g,k,l}^*$,

$$2^{d+\frac{g+k-1}{2}} \sum_{\sum a_i=d} \prod_{i=1}^l \frac{(2a_i-1)!!}{\lambda_i^{2a_i+1}} \langle \tau_{a_1} \dots \tau_{a_l} \sigma^k \rangle = \sum_{G \in \tilde{\mathcal{R}}_{g,k,l}^*} \frac{\prod_{C \in C(G)} c(C)}{|Aut(G)|} \prod_{e \in E} \lambda(e).$$

And theorem 1.6 is proved.

APPENDIX A. PROPERTIES OF THE STRATIFICATION

A.0.1. Proposition 4.23. Fix sets $\mathcal{I}, \mathcal{B}, \mathcal{P}_0$. For a stable open ribbon graph G , write $\mathcal{M}_G = \mathbb{R}_+^{E(G)}/Aut(G)$. Let $G_{g,\mathcal{B},(\mathcal{I},\mathcal{P}_0)}$ be the set of all such graphs with boundary markings, internal markings and internal markings of perimeter 0 being $\mathcal{B}, \mathcal{I}, \mathcal{P}_0$ respectively. We will show that $comb^{\mathbb{R}}$ maps $\overline{\mathcal{M}}_{g,\mathcal{B},\mathcal{I} \cup \mathcal{P}_0}^{\mathbb{R}} \rightarrow \coprod_{G_{g,\mathcal{B},(\mathcal{I},\mathcal{P}_0)}} \mathcal{M}_G(\mathbf{p})$, surjectively, and that it is 1 : 1 on smooth or effective loci.

Step 1. *An anti holomorphic involution ϱ of a connected stable curve X is separating if X/ϱ is a connected orientable stable surface with boundary. X^ϱ is called the real locus. A half of X is a stable connected subsurface with boundary $\Sigma \subseteq X$ such that the restricted map $\Sigma \rightarrow X^\varrho$ is a homeomorphism.*

A doubled $(g, B, I \cup \mathcal{P}_0)$ -surface is a closed stable marked surface X , with markings $\{x_i\}_{i \in \mathcal{B}}, \{z_i, \bar{z}_i\}_{i \in I \cup \mathcal{P}_0}$ together with a separating anti holomorphic involution ϱ and a preferred half Σ which satisfies the following

- (a) $\forall i, x_i \in X^\varrho$.
- (b) $\forall i, z_i \in \text{int}(\Sigma)$.

Observation A.1. There is a natural one to one correspondence between open stable $(g, \mathcal{B}, \mathcal{I} \cup \mathcal{P}_0)$ surfaces Σ and doubled $(g, \mathcal{B}, \mathcal{I} \cup \mathcal{P}_0)$ -surfaces (X, ϱ, Σ) , given by $\Sigma \rightarrow (D(\Sigma), \Sigma)$, where Σ is taken as a subset of $D(\Sigma)$.

Note that all components of X^ϱ which are not isolated points are canonically oriented as boundaries of the distinguished half.

Step 2. Fix positive $\{p_i\}_{i \in \mathcal{I}}$. For convenience we denote by $\bar{\mathcal{I}}, \bar{\mathcal{P}}_0$ the markings of \bar{z}_i , for $i \in \mathcal{I}, \mathcal{P}_0$. We now analyze the image of doubled surfaces (X, ϱ, Σ) under the (closed) map $\text{comb}_{\mathbf{q}}$ defined on $\bar{\mathcal{M}}_{g,k+2l}$, where the perimeters \mathbf{q} are defined so that the faces of z_i, \bar{z}_i , $i \in \mathcal{I}$ have perimeter p_i and the other points are boundary marked points or internal marked with perimeter 0. By the construction for closed surfaces, the image is a stable ribbon graph G in the sense of Definition 4.2, embedded in $\tilde{X} = K_{\mathcal{B} \cup \mathcal{P}_0 \cup \bar{\mathcal{P}}_0}(X)$. Moreover, ϱ induces an involution, which we also denote by ϱ , on \tilde{X}, G , and by Lemma 4.12, $\tilde{X}^\varrho \subseteq G$. Faces and vertices marked by $\mathcal{I} \cup \mathcal{P}_0$ are in one distinguished half, $\tilde{\Sigma}$, of \tilde{X} , where a half is defined analogously to above.

Write E^B for ϱ -invariant edges. Let H^B be their halves which do not agree with the orientation induced by $\tilde{\Sigma}$. Write V^B for ϱ -invariant vertices. Let V^I be vertices in $\text{int}(\tilde{\Sigma})$, and H^I either half edges in $s_1 H^B$ or half edges which intersect $\text{int}(\tilde{\Sigma})$, $E^I = (H^I \setminus s_1 H^B)/s_1$.

Observation A.2. s_1 leaves $H^I \cup H^B$ invariant, and that s_0 takes H^I to $H^I \cup H^B$.

Indeed, if there were $h \in H^I, h' \notin H^I \cup H^B$, with $s_0 h = h'$, then there was a common face which contained $h, s_1 h'$. But then this face would intersect both $\text{int}(\tilde{\Sigma}), \varrho(\text{int}(\tilde{\Sigma}))$, which is impossible.

Let v be a vertex, consider its half edges. The permutation s_0 acts on them, and also ϱ . Write B_v for the set of s_0 -cycles which contain an element of H^B , write I_v for those cycles in H^I . It is easy to see that no s_0 -cycle contains more than two boundary edges. It follows from the observation that inside a cycle in B_v the half edges are s_0 -ordered as h_1, \dots, h_{2r+2} so that $h_1 \in s_1 H^B$,

$$h_i \in H^I \setminus s_1 H^B, i \in [1, r+1], h_{r+2} \in H^B, h_i \notin (H^I \cup H^B), i \in [r+2, 2r+2].$$

Define a permutation \tilde{s}_0 of $H^I \cup H^B$ which is s_0 on H^I , and otherwise, we are in the scenario just describe, $\tilde{s}_0 h_{r+2} = h_1$.

Define new marking assignments, $f^{\mathcal{I}}, f^{\mathcal{B}}, f^{\mathcal{P}_0}$ as follows. $f^{\mathcal{I}}$ maps $i \in \mathcal{I}$ to the face which contains z_i , $f^{\mathcal{B}}, f^{\mathcal{P}_0}$ map $i \in \mathcal{B}$ to the vertex x_i is mapped to. $f^{\mathcal{P}_0}$ is defined similarly.

Recall Notation 4.16. Define $\tilde{D}(g, I, B)$ to be the set of isotopy classes of smooth doubled (g, I, B) -surfaces. Write $\tilde{D}(g, I) = D(g, I)$. Clearly there exists a canonical identification $\alpha : \tilde{D}(g, I, B) \simeq D(g, I, B)$.

We can enrich the graph (G, ϱ) with a defect function d on $V^I \cup V^B$ defined as follows. Let $v \in V^I \cup V^B$ be a vertex, consider its preimage

X_v in X . If X_v is not a point, then it is a pointed nodal surface, doubled in case $v \in V^B$, and otherwise just a usual closed one, without z_i, \bar{z}_i for $i \in \mathcal{I}$. Some of the special points of X_v correspond to nodes whose two halves belong to X_v . Smooth X_v along these nodes. There is a unique topological way to perform the smoothing process on a doubled surface, which is consistent with the choice of a half and such that the resulting surface is doubled. Define $d(v) \in D(g(v), I_v \cup (f^{P_0})^{-1}(v), B_v \cup (f^B)^{-1}(v))$ to be the class of the smoothen X_v in the doubled case. Otherwise $d(v)$ is the unique element in $D(g(v), I_v \cup (f^{P_0})^{-1}(v))$.

The ribbon graph G , together with the involution ϱ , and the doubled data, which is the sets H^I, H^B, V^I, V^B , and the maps d, f^I, f^B, f^{P_0} is called a doubled ribbon graph. We see that any doubled surface, together with perimeters as above, is associated with a doubled graph. Call this association $Dcomb$. It now follows from definitions that

Observation A.3. There is a canonical bijection $Half$ between doubled $(g, \mathcal{B}, (\mathcal{I}, \mathcal{P}_0))$ -metric ribbon graphs, and open $(g, \mathcal{B}, (\mathcal{I}, \mathcal{P}_0))$ -metric ribbon graphs. $Half(G)$ is the graph spanned by H^I, H^B, V^I, V^B , permutations \tilde{s}_0, s_1 , maps f^I, f^B, f^{P_0} , the same genus defect of G and topological defect $\alpha(d)$.

$Half(G)$ is embedded in $\tilde{\Sigma}$, which, after defining the corresponding defects, is exactly $K_{\mathcal{B}, \mathcal{P}_0} \Sigma$.

Thus, by Observations A.1, A.3, for any $\Sigma \in \overline{\mathcal{M}}_{g,k,l}^{\mathbb{R}}$ and perimeters \mathbf{p} , the symmetric JS differential indeed defines a stable open ribbon graph with perimeters \mathbf{p} embedded in $K_{\mathcal{B}, \mathcal{P}_0} \Sigma$.

Step 3. We now show that

Proposition A.4. $comb^{\mathbb{R}} : \overline{\mathcal{M}}_{g, \mathcal{B}, \mathcal{I} \cup \mathcal{P}_0}^{\mathbb{R}} \times \mathbb{R}^{\mathcal{I}} \rightarrow \coprod_{G_{g, \mathcal{B}, (\mathcal{I}, \mathcal{P}_0)}} \mathcal{M}_G$, is a surjection, and in the smooth case, or more generally when unmarked components are not adjacent and form a moduli of dimension 0, it is in fact a bijection on its image.

This proposition is true in the closed case. By the above construction, it will be enough to show these properties for $Dcomb$. By the closed theory, from the doubled metric graph (G, ℓ) one can reconstruct the unique surface with extra structure \tilde{X} , in which it embeds, including the complex structure on its marked components. Write \mathbf{q} for the set of perimeters of faces of G . It is evident that the perimeters of faces i, \bar{i} are the same. The involution on (G, ℓ) lifts to an involution on \tilde{X} . For any singular point $v \in \tilde{X}$, which corresponds the vertex v of the graph, any s_0 -cycle \tilde{v} of half edges corresponds a new marked point labelled \tilde{v} in the normalization of $\tilde{\Sigma}$. We define a surface X as follows.

For a singular v , if $v \in V^B$, replace v by a doubled surface Σ_v , in the isotopy class $d(v)$. For a singular $v \in V^I$, replace v , $\varrho(v)$ be two conjugate closed surfaces $\Sigma_v, \bar{\Sigma}_v$, Σ_v is in the class of $d(v)$. Note that Σ_v is not necessarily stable. Let $\Sigma_1, \dots, \Sigma_r$ be the marked components of $\tilde{\Sigma}$. Define

$$X = \text{Stab}((\coprod X_i \cup \coprod X_v) / \sim)$$

where the \sim identifies a marked point is some Σ_v which corresponds to a s_0 -cycle \tilde{v} with the corresponding point in some Σ_i . Stab is the stabilization map which contracts an unstable component to a point.

One can easily extend ϱ and the choice of an half to X , and $D\text{comb}(X, \mathbf{q}) = (G, \ell)$, where \mathbf{q} is the set of perimeters.

In the smooth or the more general case described in the statement, we have no freedom in the reconstruction of X .

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